

# INTERACTIONS BETWEEN PARA-QUATERNIONIC AND GRASSMANNIAN GEOMETRY

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**ABSTRACT.** Almost para-quaternionic structures on smooth manifolds of dimension  $2n$  are basically equivalent to almost Grassmannian structures of type  $(2, n)$ . We remind the equivalence and investigate some of its attractive consequences. In particular, we make use of Cartan-geometric techniques in treating the integrability issues on the twistor space level. The discussion includes also the so-called 0-twistor space, which turns out to be the prominent object immediately available both from the para-quaternionic and the Grassmannian point of view.

## 1. Introduction

Almost para-quaternionic structures are inspired by the algebra of para-quaternions, alike the almost quaternionic structures are based on the usual quaternions. The former and the latter structures can be seen as the split and the compact real forms of complex quaternionic structures, respectively. It is known there is a natural equivalence between the almost para-quaternionic structures on manifolds of dimension  $2n$  and the almost Grassmannian structures of type  $(2, n)$ . We plan to remind the equivalence and enjoy some of its consequences. That is to say, we plan to benefit from the normal Cartan connection associated to the almost Grassmannian structure to study some typical issues related to the almost para-quaternionic description. This in particular involves the discussion of integrability, which is directly reflected on the level of twistor spaces.

Para-quaternionic geometry in dimension 2 and 4 is quite degenerate and specific, respectively. The former case is ignored, a review of the four-dimensional case is placed in subsection 6.1. Sometimes we have to distinguish between the four- and higher-dimensional cases. In that case, the assumption on the dimension is declared explicitly.

For an almost quaternionic structure, the twistor space in the sense of Salamon [16] is the bundle of all (almost) complex structures within the almost quaternionic structure under consideration. The total space of that bundle carries a canonical almost complex structure which turns out to be integrable if and only if the underlying almost quaternionic structure is integrable. However, in the case of almost para-quaternionic structures there are three types of twistor spaces which are distinguished by the sign  $\epsilon \in \{-1, 0, 1\}$ : The  $\epsilon$ -twistor space is the bundle consisting of the so-called (almost)  $\epsilon$ -complex structures, which are contained in the almost para-quaternionic structure. The almost  $\epsilon$ -complex structure is an endomorphism of the tangent bundle, which squares to the  $\epsilon$ -multiple of the identity map. Classically,

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$(-1)$ -,  $0$ -, and  $1$ -complex structures are called complex, tangent, and para-complex structures, respectively. It is a known fact there is a canonical almost  $\epsilon$ -complex structure on each  $\epsilon$ -twistor space, where  $\epsilon = \pm 1$ , such that its integrability is equivalent to the integrability of the underlying almost para-quaternionic structure [2, 9]. We provide an alternative control of this fact and extend the discussion also to the case  $\epsilon = 0$ , see subsection 4.5.

The Salamon's construction mentioned above is an attempt to generalize the Penrose's twistor theory (for 4-dimensional anti-self-dual conformal structures) introduced in [15] to higher dimensional cases. Bailey and Eastwood presents in [4] another generalization of that correspondence to higher dimensions. The adaptation of the former approach to almost para-quaternionic case has just been described in the previous paragraph, while the real version of the latter approach yields a twistor theory for almost Grassmannian structures, cf. [13]. Hence, focusing on almost Grassmannian structures of type  $(2, n)$ , which are the same as almost para-quaternionic structures in dimension  $2n$ , we have got two natural but different twistor constructions in hand. It is a natural question whether these two constructions have anything in common, apart from their roots. We show it is just the  $0$ -twistor space, which coincides with the vertical subbundle in a natural circle bundle (the so-called correspondence space) associated to the almost Grassmannian structure, see subsection 4.6.

In sections 2 and 3 we survey those notions from para-quaternionic and Grassmannian geometry which are relevant to our aims. Note that both almost para-quaternionic and almost Grassmannian structures are called by various names in various sources; the most often synonyms are remarked in the respective subsections. Several original observations can be found in subsequent two sections. Section 4 collects all the interactions mentioned above, few more remarks valid in the integrable case are in section 5. Namely, it is another interpretation of the  $0$ -twistor space and a comment on a specific holonomy reduction firstly established by Bryant in [5], later discussed in [6, 14]. Final remarks on the specific four-dimensional case and compatible metric issues are placed in section 6. An overview of standard and often used facts from the theory of Cartan and parabolic geometries are gathered in Appendix A.

The main intent of this article is to use consistently the equivalence between almost para-quaternionic and almost Grassmannian structures of the considered type, which is otherwise only implicitly present in the literature. As we already emphasized, the techniques we use are heavily based on the equivalent description of almost Grassmannian structures as Cartan (parabolic) geometries. Of course, the construction of a normal Cartan connection to a given geometric structure is quite an involved process. However, once the work is done, then vast and often ad hoc computations may be replaced by perfectly conceptual arguments. That is why we believe the current approach offers a new and possibly efficient perspective. In this respect, the influence of the monograph [7] by Čap and Slovák should be evident. That book, especially its fourth chapter, was the main source of inspiration for this work.

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## 2. Almost para-quaternionic structures

After a remembrance of para-quaternions, we describe the almost para-quaternionic structures and the related twistor constructions. The basic reference for this section is [2] and the survey article [3].

**2.1. Para-quaternions.** The algebra of *para-quaternions*, denoted by  $\mathbb{H}_s$ , is characterized as the unique 4-dimensional real associative algebra with *indefinite* multiplicative norm. (Instead of the prefix para-, various synonyms can be found in the literature; split- is probably the most often.) Para-quaternions are written as  $q = a + bi + cj + dk$ , where the defining relations are

$$i^2 = j^2 = 1 \text{ and } k = ij = -ji; \quad (1)$$

consequently,  $k^2 = -1$ ,  $ik = -ki = j$ , and  $jk = -kj = -i$ . The conjugate para-quaternion to  $q$  is  $\bar{q} = a - bi - cj - dk$  and the multiplicative norm  $|q|^2$  is given by  $|q|^2 = q\bar{q} = \bar{q}q$ ; the corresponding polar form is  $\langle p, q \rangle = \text{Re}(p\bar{q})$ . Purely imaginary para-quaternions are characterized by  $\bar{q} = -q$ , hence  $q^2 = -q\bar{q} = -|q|^2$  holds for any  $q \in \text{Im } \mathbb{H}_s$ . The quadruple  $(1, i, j, k)$  forms an orthonormal basis of  $\mathbb{H}_s$ , where  $|i|^2 = |j|^2 = -1$  and  $|k|^2 = 1$ . There are two-dimensional isotropic subspaces in  $\mathbb{H}_s$ , hence the inner product has got the split signature  $(2, 2)$ .

The algebra of para-quaternions is isomorphic to the algebra of endomorphism of  $\mathbb{R}^2$ , i.e. the matrix algebra  $\text{Mat}_{2 \times 2}(\mathbb{R})$ , such that the norm squared corresponds to the determinant. The isomorphism is given by

$$a + bi + cj + dk \mapsto \begin{pmatrix} a+b & c+d \\ c-d & a-b \end{pmatrix}, \quad (2)$$

in particular, the standard basis is mapped as follows:

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3)$$

In these terms, the purely imaginary para-quaternions  $\text{Im } \mathbb{H}_s \subset \mathbb{H}_s$  form the three-dimensional subspace of trace-free matrices. In particular, it is an invariant subspace under the conjugation by any regular matrix.

The group of automorphisms of  $\mathbb{H}_s$  is just the subgroup of those elements of  $SO(\mathbb{H}_s) \cong SO(2, 2)$  which acts on  $\text{Re } \mathbb{H}_s$  by the identity. Hence  $\text{Aut}(\mathbb{H}_s)$  is isomorphic to  $SO_0(1, 2)$ , the connected component of the identity element in  $SO(1, 2)$ . Under the identification above, the group of unit para-quaternions  $\{q \in \mathbb{H}_s : |q|^2 = 1\}$  is isomorphic to  $SL(2, \mathbb{R})$ . The conjugation by any such element,  $p \mapsto qpq^{-1} = qp\bar{q}$ , yields a surjective group homomorphism  $SL(2, \mathbb{R}) \rightarrow \text{Aut}(\mathbb{H}_s)$  whose kernel is  $\{\pm 1\}$ . This just recovers the two-fold covering  $SL(2, \mathbb{R}) \rightarrow PSL(2, \mathbb{R})$  or, isomorphically written,  $\text{Spin}(1, 2) \rightarrow SO_0(1, 2)$ .

**2.2. Almost para-quaternionic structures.** An *almost para-quaternionic structure* on a smooth manifold  $M$  of even dimension  $2n \geq 4$ , is given by a subbundle  $\mathcal{Q} \subset \text{End}(TM)$  of rank 3, which is (locally) generated by  $I, J, K$  such that

$$I \circ I = J \circ J = \text{id} \text{ and } K = I \circ J = -J \circ I; \quad (4)$$

consequently,  $K \circ K = -\text{id}$ ,  $I \circ K = -K \circ I = J$ , and  $J \circ K = -K \circ J = -I$ . The triple  $(I, J, K)$  is called a (local) basis of  $\mathcal{Q}$ . A compatible affine connection, the so-called *para-quaternionic connection*, is a linear connection on  $TM$  which preserves the subbundle  $\mathcal{Q} \subset \text{End}(TM)$ . An almost para-quaternionic structure is

called *para-quaternionic*, or *integrable*, if there is a para-quaternionic connection which is torsion-free.

The bundle  $\mathcal{Q}$  is endowed with a bundle metric of signature  $(1, 2)$ , determined by

$$A \circ A = -|A|^2 \text{id}, \quad (5)$$

and an orientation, determined by any (local) basis of  $\mathcal{Q}$ . In particular, the typical fiber of  $\mathcal{Q} \rightarrow M$  is the standard oriented Minkowski space. In the terminology of [2, 8], elements  $A \in \mathcal{Q}$  such that  $|A|^2$  is 1, 0, and  $-1$  (i.e. endomorphisms such that  $A \circ A$  is  $-\text{id}$ , 0, and  $\text{id}$ ), are called *almost complex*, *almost tangent*, and *almost para-complex* structures, respectively. Below we mostly use the uniform abbreviation *almost  $\epsilon$ -complex structure*, where  $\epsilon = -|A|^2 \in \{-1, 0, 1\}$ . We use the same terminology and notation also on the vector space level.

Note that, for an almost tangent structure  $A \in \mathcal{Q}$ , the condition  $A \circ A = 0$  implies  $\text{im } A = \ker A$ , which yields a distinguished rank  $n$  distribution in  $TM$ . Similarly, for a para-complex structure  $A \in \mathcal{Q}$ , the  $\pm 1$ -eigenspace decomposition of  $TM$  forms two complementary distributions in  $TM$  of the same rank  $n$ .

**Remarks.** (1) Note that almost para-quaternionic manifolds may indeed be of any even dimension. Only the existence of a non-degenerate compatible metric brings an additional restriction so that the dimension of the base manifold has to be a multiple of four. See 6.2 for some details.

(2) Almost para-quaternionic structures can be regarded as G-structures so that the notion of integrability above is the usual notion of 1-integrability for G-structures, see e.g. [2, 9]. The corresponding structure group is discussed in 4.1.

(3) Almost para-quaternionic structures appear under different names in the literature, the frequent term in older references is almost quaternionic structures of second type. There are also alternative definitions of the structure, see e.g. [17] for more informations.

**2.3. Twistor spaces for almost para-quaternionic manifolds.** Given an almost para-quaternionic manifold  $(M, \mathcal{Q})$  and an arbitrary  $s \in \mathbb{R}$ , the *s-twistor space*  $\mathcal{Z}^s \rightarrow M$  is defined as

$$\mathcal{Z}^s := \{A \in \mathcal{Q} : |A|^2 = -s, \text{ i.e. } A \circ A = s \text{id}\}.$$

By definition, each *s*-twistor space is a fibre bundle over  $M$  with 2-dimensional fibre, hence the dimension of the total space is also even.

The typical fiber of  $\mathcal{Q} \rightarrow M$  is decomposed into disjoint subsets consisting of space-, light-, and time-like vectors. Accordingly we denote the decomposition of  $\mathcal{Q}$  by  $\mathcal{Q} = \mathcal{Q}^+ \sqcup \mathcal{Q}^0 \sqcup \mathcal{Q}^-$ . For  $s < 0$  the typical fibers of  $\mathcal{Z}^s \rightarrow M$  are hyperboloids of two sheets, which are mutually identified via the central projection. Similarly for  $s > 0$ , where these are hyperboloids of one sheet. Hence for any  $s > 0$  and  $s < 0$ , the *s*-twistor space  $\mathcal{Z}^s$  is identified with the projectivization  $\mathcal{P}\mathcal{Q}^+$  and  $\mathcal{P}\mathcal{Q}^-$ , respectively, and we use the following notation:

$$\mathcal{Z}^\pm := \mathcal{Z}^{\pm 1} \cong \mathcal{P}\mathcal{Q}^\pm.$$

However, for  $s = 0$ , the typical fiber is the cone of null-vectors. Hence

$$\mathcal{Z}^0 = \mathcal{Q}^0$$

and its projectivization is a circle bundle over  $M$  which will play a distinguished role later on. Altogether, we consider just three types of  $s$ -twistor spaces distinguished by the sign of  $s$ .

The following statement is formulated as Proposition 6 in [2]:

**Proposition.** *Let  $(M, \mathcal{Q})$  be an almost para-quaternionic manifold and let  $\epsilon \in \{-1, 0, 1\}$ . Any para-quaternionic connection induces a natural almost  $\epsilon$ -complex structure  $\mathcal{J}^\epsilon$  on the  $\epsilon$ -twistor space  $\mathcal{Z}^\epsilon$ .*

**Remarks.** (1) Roughly, the construction is as follows. A para-quaternionic connection  $\nabla$  gives rise to a horizontal distribution  $H^\nabla \subset T\mathcal{Z}^\epsilon$ , complementary to the vertical subbundle of the projection  $p : \mathcal{Z}^\epsilon \rightarrow M$ . Firstly, the vertical subspace at any  $z \in \mathcal{Z}^\epsilon$  is identified with the tangent space of an appropriate quadric in the (oriented) Minkowski space, hence it carries a canonical  $\epsilon$ -complex structure. Secondly, any  $z \in \mathcal{Z}^\epsilon$  defines an almost  $\epsilon$ -complex structure in  $T_{p(z)}M$ , which is lifted up to  $H_z^\nabla$  via the inverse map of  $T_z p$ . The two pieces then assemble to a natural almost  $\epsilon$ -complex structure on  $T_z \mathcal{Z}^\epsilon$ .

Sections of the bundle projection  $p : \mathcal{Z}^\epsilon \rightarrow M$  are by definition almost  $\epsilon$ -complex structures on  $M$ . Denoting by  $J^s$  the  $\epsilon$ -complex structure corresponding to the section  $s : M \rightarrow \mathcal{Z}^\epsilon$ , the construction above yields also the following relation:

$$J^s = Tp \circ \mathcal{J}^\epsilon \circ Ts. \quad (6)$$

(2) In [9] it is shown the induced almost  $\epsilon$ -complex structure on the  $\epsilon$ -twistor space does not actually depend on the choice of the para-quaternionic connection provided the connection is *minimal*, which means it has the “minimal” torsion determined by the corresponding G-structure. The almost  $\epsilon$ -complex structure induced by a minimal para-quaternionic connection is hereinafter called *canonical*.

**2.4. Integrability.** The main issue of the previous construction is that the integrability of the almost para-quaternionic structure is fully controlled by the integrability of the canonical almost  $\epsilon$ -complex structures on the twistor spaces. The following statement is extracted from Theorem 21 in [9] (cf. also Theorem 2 in [2]):

**Theorem.** *Let  $(M, \mathcal{Q})$  be an almost para-quaternionic manifold of dimension  $2n > 4$ . Let  $(\mathcal{Z}^\epsilon, \mathcal{J}^\epsilon)$  be the  $\epsilon$ -twistor space with the canonical almost  $\epsilon$ -complex structure, where  $\epsilon = \pm 1$ . Then  $\mathcal{Q}$  is integrable if and only if  $\mathcal{J}^\epsilon$  is integrable.*

**Remark.** Integrability of the  $\epsilon$ -complex structure in the previous statement is by definition equivalent to the vanishing of the Nijenhuis tensor. Given a smooth manifold  $Z$  and an endomorphism  $A \in \text{End}(TZ) = \Omega^1(Z, TZ)$ , the *Nijenhuis tensor* of  $A$  is given by the Frölicher–Nijenhuis bracket:  $N_A := \frac{1}{2}[A, A] \in \Omega^2(Z, TZ)$ , i.e.

$$N_A(\xi, \eta) := -A^2[\xi, \eta] - [A\xi, A\eta] + A[A\xi, \eta] + A[\xi, A\eta], \quad (7)$$

for  $\xi, \eta \in TZ$ .

Note that for almost para-complex structures, the Nijenhuis tensor vanishes if and only if the corresponding distributions are integrable in the sense of Frobenius. However, for almost tangent structures, vanishing of the Nijenhuis tensor is stronger than the integrability of the corresponding distribution. I.e. the integrability of  $A$  implies the integrability of  $\text{im } A = \ker A$ , while the other implication need not hold true, cf. e.g. [12].

### 3. Almost Grassmannian structures

Here we collect several equivalent definitions of almost Grassmannian structures. In particular, we emphasize the existence of the normal Cartan connection and the corresponding correspondence/twistor space construction. Relevant classical references are [1, 4, 13].

**3.1. Almost Grassmannian structures.** An *almost Grassmannian structure* of type  $(p, q)$  on a smooth manifold of dimension  $pq$  is given by an identification of the tangent bundle with the tensor product of two auxiliary vector bundles of ranks  $p$  and  $q$ . A compatible affine connection is a linear connection on the tangent bundle, which is the tensor product of two linear connections on the auxiliary vector bundles. An almost Grassmannian structure is called *Grassmannian*, or *integrable*, if there is a compatible affine connection which is torsion-free.

The model Grassmannian structure is observed on  $\text{Gr}_p(\mathbb{R}^{p+q})$ , the Grassmannian of  $p$ -planes in the  $(p+q)$ -dimensional vector space. Indeed, the tangent space at any  $\lambda \in \text{Gr}_p(\mathbb{R}^{p+q})$  is naturally identified with  $\lambda^* \otimes (\mathbb{R}^{p+q}/\lambda)$ , the space of linear maps from  $\lambda$  to the factor space.

**Convention.** Throughout this paper we consider almost Grassmannian structures just of type  $(2, n)$ , where  $n \geq 2$ . Let us denote the auxiliary vector bundles of rank 2 and  $n$  by  $E \rightarrow M$  and  $F \rightarrow M$ , respectively. Almost Grassmannian structure of type  $(2, n)$  on  $M$  is then given by an isomorphism

$$E^* \otimes F \xrightarrow{\cong} TM.$$

**Remarks.** (1) As a part of the definition, it is often taken an additional identification

$$\Lambda^2 E^* \cong \Lambda^n F$$

or, equivalently, a trivialisation of the line bundle  $\Lambda^2 E \otimes \Lambda^n F$ . This just brings the notion of orientation into play; the corresponding geometric structure is called the *oriented* almost Grassmannian structure of type  $(2, n)$ . The model in this case is the Grassmannian of oriented 2-planes in  $\mathbb{R}^{2+n}$  rather than the ordinary one.

(2) Seeing an almost Grassmannian structure as a G-structure, the structure group is isomorphic to

$$GL(2, \mathbb{R}) \otimes GL(n, \mathbb{R}) \cong GL(2, \mathbb{R}) \cdot GL(n, \mathbb{R}), \quad (8)$$

the quotient of  $GL(2, \mathbb{R}) \times GL(n, \mathbb{R})$  by the center, which consists of real multiples of the identity. The structure group for oriented Grassmannian structures is the lift  $S(GL(2, \mathbb{R}) \times GL(n, \mathbb{R})) \subset GL(2, \mathbb{R}) \times GL(n, \mathbb{R})$ . In these terms, the above notion of integrability is the usual 1-integrability for G-structures.

(3) Almost Grassmannian (or closely related) structures appear under various names in the literature, e.g. as paraconformal structures in [4] or Segre structures in [1, 10]. In the latter manner, the structure is introduced as a field of *Segre cones*. The Segre cone in  $T_x M \cong E_x^* \otimes F_x$  is exactly the set of all linear maps  $E_x \rightarrow F_x$  which are of the rank one. The Segre cone is doubly ruled by linear subspaces of dimensions 2 and  $n$ . We denote by  $\mathcal{F}$  and  $\mathcal{E}$  the subbundle in  $\text{Gr}_2(TM)$  and  $\text{Gr}_n(TM)$  consisting of all the 2- and  $n$ -dimensional subspaces in  $TM$ , respectively, which belong to the Segre structure. Elements  $V \in \mathcal{F}_x$  are parametrized by 1-dimensional subspaces  $v \in F_x$  such that  $V = \{\xi : E_x \rightarrow F_x : \text{im } \xi = v\}$ . Similarly, elements  $U \in \mathcal{E}_x$  are parametrized by 1-dimensional subspaces  $u \in E_x$  such that

$U = \{\xi : E_x \rightarrow F_x : \ker \xi = u\}$ . Hence we have identified  $\mathcal{F} \cong \mathcal{P}F$  and  $\mathcal{E} \cong \mathcal{P}E$ , in particular, the typical fibers of the respective bundles over  $M$  are  $\mathbb{RP}^{n-1}$  and  $\mathbb{RP}^1$ . Elements of  $\mathcal{F}$  and  $\mathcal{E}$  are the so-called  $\alpha$ - and  $\beta$ -planes, respectively. An almost Grassmannian structure is called  $\beta$ -semi-integrable, if any  $\beta$ -plane from  $\mathcal{E}$  is tangent to a unique immersed  $n$ -dimensional submanifold of  $M$  whose all tangent spaces are elements of  $\mathcal{E}$ . The notion of  $\alpha$ -semi-integrability is defined similarly.

**3.2. Normal Cartan connection.** The Grassmannian  $\text{Gr}_2(\mathbb{R}^{2+n})$  is the homogeneous space with the obvious transitive action of the Lie group  $G := \text{PGL}(2+n, \mathbb{R})$ . Denoting by  $P$  the stabilizer of a 2-plane in  $\mathbb{R}^{2+n}$ , we have got  $\text{Gr}_2(\mathbb{R}^{2+n}) \cong G/P$ . If the 2-plane is  $\langle e_1, e_2 \rangle$ , the linear span of the first two vectors of the standard basis of  $\mathbb{R}^{2+n}$ , then  $P$  is represented by the block triangular matrices

$$\left( \begin{array}{c|c} A & Z \\ \hline 0 & B \end{array} \right)$$

with the blocks of sizes 2 and  $n$  along the diagonal. The subgroup  $P \subset G$  is parabolic and the Levi subgroup in  $P$ , denoted by  $G_0$ , is represented by the block diagonal matrices of the above form (i.e. with  $Z = 0$ ). Evidently,

$$G_0 = P(GL(2, \mathbb{R}) \times GL(n, \mathbb{R})),$$

which is just the structure group displayed in (8).

The parabolic subgroup  $P \subset G$  induces a grading of the Lie algebra  $\mathfrak{g} = \mathfrak{sl}(2+n, \mathbb{R})$ , which is displayed in the following block decomposition

$$\left( \begin{array}{c|c} \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array} \right)$$

with blocks of sizes 2 and  $n$ . In particular,  $\mathfrak{g}_{-1} \cong \mathbb{R}^{2*} \otimes \mathbb{R}^n$ ,  $\mathfrak{g}_0 \cong \mathfrak{sl}(\mathfrak{gl}(2, \mathbb{R}) \times \mathfrak{gl}(n, \mathbb{R}))$ , and  $\mathfrak{g}_1 \cong \mathbb{R}^2 \otimes \mathbb{R}^{n*}$ . The semi-simple part of  $\mathfrak{g}_0$  is  $\mathfrak{g}_0^{ss} \cong \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(n, \mathbb{R})$ .

Now it is a very important fact for what follows that almost Grassmannian structures can be described as parabolic geometries of the current type. For details and the proofs of following statements see [13] or subsection 4.1.3. in [7]:

**Proposition.** *An almost Grassmannian structure of type  $(2, n)$  on  $M$  is equivalent to a normal parabolic geometry  $(\mathcal{G} \rightarrow M, \omega)$  of type  $G/P$ , where  $G = \text{PGL}(2+n, \mathbb{R})$  and  $P$  is the parabolic subgroup as above. In these terms, the integrability of the structure is equivalent to the vanishing of the torsion of the Cartan connection, which is automatically satisfied for  $n = 2$ .*

**Remarks.** (1) Considering oriented almost Grassmannian structures, the previous characterization is very similar up to the different choice of the Lie group to the Lie algebra  $\mathfrak{g} = \mathfrak{sl}(2+n, \mathbb{R})$ . Namely, it would be  $G = \text{SL}(2+n, \mathbb{R})$  rather than  $\text{PGL}(2+n, \mathbb{R})$ , which reflects the interpretation of the Grassmannian of oriented 2-planes in  $\mathbb{R}^{2+n}$  as a homogeneous space.

(2) The notion of normality in the previous Proposition is the standard normalization condition for parabolic geometries, see A.2(3). In the case  $n > 2$ , the corresponding harmonic curvature has got one component of homogeneity 1 and one component of homogeneity 2, which we denote by  $T$  and  $K$ , respectively. A finer description of the respective components of  $\ker \square \subset \Lambda^2 \mathfrak{g}_1 \otimes \mathfrak{g}$  is written in the table below. The actual irreducible component belonging to  $\ker \square$  is always the highest weight part contained in the indicated representation. In particular, it lies in the kernel of all possible traces.

homog.	contained in representation
1	$S^2\mathbb{R}^2 \otimes \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^{2*} \otimes \mathbb{R}^n$
2	$\Lambda^2\mathbb{R}^2 \otimes S^2\mathbb{R}^{n*} \otimes \mathfrak{sl}(n, \mathbb{R})$

The individual harmonic curvature components may then be seen as sections of the following tensor bundles:

$$T \in \Gamma(S^2E \otimes \Lambda^2F^* \otimes E^* \otimes F), \quad K \in \Gamma(\Lambda^2E \otimes S^2F^* \otimes \mathfrak{sl}(F)).$$

In the case  $n = 2$ , there are two components of homogeneity 2, which we denote by  $K_1$  and  $K_2$ . It turns out they are sections of the following tensor bundles:

$$K_1 \in \Gamma(S^2E \otimes \Lambda^2F^* \otimes \mathfrak{sl}(E)), \quad K_2 \in \Gamma(\Lambda^2E \otimes S^2F^* \otimes \mathfrak{sl}(F)).$$

(3) In these terms, the  $\beta$ -semi-integrability discussed in remark 3.1(3) is equivalent to the vanishing of  $T$  and  $K_1$ , providing that  $n > 2$  and  $n = 2$ , respectively.

(4) Parabolic geometries of the current type are  $|1|$ -graded and so the lowest homogeneous curvature components are of degree 1. Hence, by A.2(4), the torsion  $\tau$  of the normal Cartan connection coincides with the harmonic torsion. It further turns out this torsion coincides with the torsion of any underlying Weyl connection. From the description above we see it is just the tensor  $T$  if  $n > 2$ , while it vanishes identically if  $n = 2$ . In particular, the integrability of the structure is equivalent to the  $\beta$ -semi-integrability if  $n > 2$ , while it is automatically satisfied if  $n = 2$ .

**3.3. Twistor correspondence for almost Grassmannian structures.** Let  $G = PGL(2+n, \mathbb{R})$  and  $P \subset G$  be as above. Let  $P' \subset G$  be the stabilizer of the line  $\langle e_1 \rangle$  spanned by the first vector of the standard basis in  $\mathbb{R}^{2+n}$ . Hence  $Q := P \cap P'$  is the stabilizer of the flag  $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle$ . Alike  $G/P \cong \text{Gr}_2(\mathbb{R}^{2+n})$ , the homogeneous space  $G/P'$  is identified with the projective space  $\mathbb{RP}^{1+n}$ , and  $G/Q$  with the proper flag manifold. The flag manifold  $G/Q$  is fibered both over the Grassmannian  $G/P$  and over the projective space  $G/P'$ . Since it provides a kind of correspondence between  $G/P$  and  $G/P'$ , it is called the *correspondence space* of  $G/P$  and  $G/P'$ , while the latter spaces are the *twistor spaces* of  $G/Q$ . For later use, let us figure the respective subgroups of  $G$  in the block matrix form:

$$Q = \left( \begin{array}{cc|c} a & b & Z_1 \\ 0 & d & Z_2 \\ \hline 0 & 0 & B \end{array} \right),$$

$$P = \left( \begin{array}{cc|c} a & b & Z_1 \\ c & d & Z_2 \\ \hline 0 & 0 & B \end{array} \right), \quad P' = \left( \begin{array}{cc|c} a & b & Z_1 \\ 0 & d & Z_2 \\ \hline 0 & Y & B \end{array} \right),$$

where the separators distinguish the blocks of sizes 2 and  $n$  as before; in particular  $a, b, c, d \in \mathbb{R}$ .

In general, for the normal parabolic geometry  $(\mathcal{G} \rightarrow M, \omega)$  of type  $G/P$  associated to an almost Grassmannian structure on  $M$ , the *correspondence space* of  $M$  with respect to  $Q \subset P$  is the orbit space

$$\mathcal{CM} := \mathcal{G}/Q.$$

It is the total space of the fibre bundle  $\mathcal{CM} \rightarrow M$  whose typical fiber is  $P/Q \cong \mathbb{RP}^1$ . The restricted Cartan geometry  $(\mathcal{G} \rightarrow \mathcal{CM}, \omega)$  is a parabolic geometry of type  $G/Q$ . Regular and normal parabolic geometries of this type are equivalent to the so-called *generalized path geometries*. Generalized path geometry on  $\mathcal{CM}$  consists of

two subbundles  $D, V \subset TCM$  of rank 1 and  $n$ , respectively, with trivial intersection and some other properties; see [10] or subsection 4.4.3 in [7]. Under the identification  $TCM \cong \mathcal{G} \times_Q (\mathfrak{g}/\mathfrak{q})$ , the two subbundles are

$$D \cong \mathcal{G} \times_Q (\mathfrak{p}/\mathfrak{q}), \quad V \cong \mathcal{G} \times_Q (\mathfrak{p}'/\mathfrak{q}), \quad (9)$$

where  $\mathfrak{q}$ ,  $\mathfrak{p}$ , and  $\mathfrak{p}'$  is the Lie algebra to  $Q$ ,  $P$ , and  $P'$ , respectively. The following statement is the subject of Proposition 4.4.5 in [7]:

**Proposition.** *Let  $E^* \otimes F \xrightarrow{\cong} TM$  be an almost Grassmannian structure of type  $(2, n)$  on  $M$  and let  $(\mathcal{G} \rightarrow M, \omega)$  be the corresponding normal parabolic geometry.*

- (1) *The correspondence space  $\mathcal{CM}$  is identified with  $\mathcal{PE}$ , the projectivization of the vector bundle  $E \rightarrow M$  of rank 2.*
- (2) *The normal parabolic geometry  $(\mathcal{G} \rightarrow M, \omega)$  is torsion-free (i.e. the almost Grassmannian structure on  $M$  is integrable) if and only if the normal parabolic geometry  $(\mathcal{G} \rightarrow \mathcal{CM}, \omega)$  over the correspondence space is regular.*
- (3) *The torsion of the latter Cartan geometry vanishes identically, while its curvature is a lift of the Grassmannian curvature.*

**Remark.** (1) Immediately by its definition, the line subbundle  $D \subset TCM$  is just the vertical subbundle of the projection  $\mathcal{CM} \rightarrow M$ . The meaning of the other subbundle  $V \subset TCM$  is revealed in the torsion-free case in 5.1.

(2) The identification  $\mathcal{PE} \cong \mathcal{CM}$  in the previous Proposition is visible as follows: The vector bundle  $E \rightarrow M$  corresponds to the standard representation of (the left-upper block of)  $P$  on  $\mathbb{R}^2$ . The stabilizer of the line spanned by the first basis vector is just the subgroup  $Q \subset P$ . This means the projectivization  $\mathcal{PE}$  is identified with  $\mathcal{G} \times_P (P/Q) = \mathcal{CM}$ .

#### 4. The interactions

Here we finally present some interactions. Firstly we discuss the natural equivalence between almost para-quaternionic structures and almost Grassmannian structures of type  $(2, n)$ . As an immediate consequence, we associate a canonical normal Cartan connection to any almost para-quaternionic structure. This allows an alternative interpretation of the twistor constructions from 2.3 involving a convenient control of integrability. At the end we uncover the role of the 0-twistor spaces, which provide a natural link between the two concepts of the twistor construction mentioned in 2.3 and 3.3.

**4.1. Equivalence of structures.** As we mentioned in remarks 2.2(2) and 3.1(2), both almost para-quaternionic structures and almost Grassmannian structures of type  $(2, n)$  can be regarded as G-structures. This way, the equivalence can be exhibited by identifying the corresponding structure groups. In order to describe the structure groups one always passes to the vector space level  $T_x M \cong \mathbb{R}^{2n}$  such that the structure group is the subgroup of  $GL(T_x M) \cong GL(2n, \mathbb{R})$ . However, at this stage, the equivalence of structures reveals immediately (cf. subsection 4.3 in [2]):

Consider we are given a linear isomorphism  $\mathbb{R}^{2n} \cong \mathbb{R}^{2*} \otimes \mathbb{R}^n$ . Then an endomorphism of  $\mathbb{R}^2$  gives rise to an endomorphism of  $\mathbb{R}^{2n}$  via the action on the first factor. The restriction just to the trace-free endomorphisms of  $\mathbb{R}^2$  yields a 3-dimensional subspace in  $\text{End}(\mathbb{R}^{2n})$ . This obviously defines a para-quaternionic structure on the vector space  $\mathbb{R}^{2*} \otimes \mathbb{R}^n \cong \mathbb{R}^{2n}$ , which we call the *standard para-quaternionic structure* and denote by  $\mathcal{Q}_{std}$ .

Conversely, given a para-quaternionic structure  $\mathcal{Q}_{\mathbb{R}^{2n}}$  on  $\mathbb{R}^{2n}$ , the algebra  $\langle \text{id} \rangle + \mathcal{Q}_{\mathbb{R}^{2n}}$  is isomorphic to the algebra of para-quaternions, i.e. to the matrix algebra  $\text{Mat}_{2 \times 2}(\mathbb{R})$ . Any irreducible  $\text{Mat}_{2 \times 2}(\mathbb{R})$ -module is isomorphic to  $\mathbb{R}^2$ , hence the  $\text{Mat}_{2 \times 2}(\mathbb{R})$ -module  $\mathbb{R}^{2n}$  is isomorphic to the tensor product  $\mathbb{R}^2 \otimes \mathbb{R}^n$ . Under this identification, the action of  $\mathcal{Q}_{\mathbb{R}^{2n}}$  on  $\mathbb{R}^{2n}$  corresponds to the action on the first factor.

**Remark.** For later use, we add some details to the notion of standard para-quaternionic structure  $\mathcal{Q}_{std}$ . Elements of  $\mathbb{R}^{2*} \otimes \mathbb{R}^n$  are seen as linear maps  $X : \mathbb{R}^2 \rightarrow \mathbb{R}^n$ . For a trace-free endomorphism  $\underline{A}$  of  $\mathbb{R}^2$ , the corresponding element  $A$  of  $\mathcal{Q}_{std}$ , i.e. an endomorphism of  $\mathbb{R}^{2*} \otimes \mathbb{R}^n$ , is given by:

$$A(X) = X \circ \underline{A}. \quad (10)$$

In these terms, the norm squared on  $\mathcal{Q}_{std}$ , defined by (5), corresponds just to the determinant:

$$|A|^2 = \det \underline{A} \quad (11)$$

**Lemma.** *Let  $\mathcal{Q}_{std}$  be the standard para-quaternionic structure on the vector space  $\mathbb{R}^{2*} \otimes \mathbb{R}^n$ . Then a non-zero element of  $\mathbb{R}^{2*} \otimes \mathbb{R}^n$  belongs to the Segre cone if and only if it is an eigenvector of a para-complex structure (equivalently, of a tangent structure) from  $\mathcal{Q}_{std}$ .*

*Proof.* With the previous notation,  $X : \mathbb{R}^2 \rightarrow \mathbb{R}^n$  belongs to the Segre cone if and only if the kernel of  $X$  has dimension one. Arbitrary complementary subspace  $\ell$  to  $\ker X$  in  $\mathbb{R}^2$  determines a para-complex structure  $\underline{A}$  such that  $\ell$  and  $\ker X$  are its eigenspaces corresponding to 1 and  $-1$ , respectively. Then obviously  $X$  is an eigenvector (corresponding to the eigenvalue 1) of the associated para-complex structure  $A$  on  $\mathbb{R}^{2*} \otimes \mathbb{R}^n$  according to (10).

Conversely, let  $X$  be an  $+1$ -eigenvector of a para-complex structure  $A \in \mathcal{Q}_{std}$  and let  $\underline{A}$  be the corresponding para-complex structure on  $\mathbb{R}^2$ ; i.e.  $A(X) = X \circ \underline{A} = X$ . Since  $\underline{A}$  is not the identity,  $X$  cannot be of full rank and, since  $X \neq 0$ , it has got rank one. Hence  $X$  belongs to the Segre cone.

The equivalent characterization in terms of tangent structures instead of para-complex ones is obvious.  $\square$

In fact, from the previous proof it also follows the eigenspaces of para-complex structures (equivalently, the kernels of tangent structures) from  $\mathcal{Q}_{std}$  form the maximal linear subspaces contained in the Segre cone of  $\mathbb{R}^{2*} \otimes \mathbb{R}^n$ . Altogether, the main issue of this subsection may be stated as follows.

**Corollaries.** *Let  $M$  be a smooth manifold  $M$  of dimension  $2n \geq 4$ .*

- (1) *There is a natural bijective correspondence between (equivalent classes of) almost Grassmannian structures  $E^* \otimes F \xrightarrow{\cong} TM$  of type  $(2, n)$  and almost para-quaternionic structures  $\mathcal{Q} \subset \text{End}(TM)$ .*
- (2) *Under this identification, the maximal linear subspaces contained in the Segre cone in  $TM \cong E^* \otimes F$ , the  $\beta$ -planes, are just the eigenspaces of para-complex structures (equivalently, the kernels of almost tangent structures) from the corresponding  $\mathcal{Q} \subset \text{End}(TM)$ .*

**4.2. General setup.** Let  $(M, \mathcal{Q})$  be an almost para-quaternionic manifold. Seeing this as an almost Grassmannian structure, let  $(\mathcal{G} \rightarrow M, \omega)$  be the induced normal parabolic geometry of type  $G/P$ . For any  $x \in M$ , the Cartan connection  $\omega$  identifies  $(T_x M, \mathcal{Q}_x)$  with  $(\mathfrak{g}_{-1}, \mathcal{Q}_{std})$ , where  $\mathcal{Q}_{std}$  denotes the standard para-quaternionic structure on  $\mathfrak{g}_{-1} \cong \mathbb{R}^{2*} \otimes \mathbb{R}^n$ . This is determined by trace-free endomorphisms of  $\mathbb{R}^2$ , which are represented by the left-upper block from matrix description of  $\mathfrak{g}_0^{ss} \cong \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(n, \mathbb{R})$  in 3.2.

The parabolic subgroup  $P$  acts on  $\mathcal{Q}_{std}$  via the adjoint action so that the orbits of the action consist of those elements, which have got the same norm. For any  $\epsilon \in \{-1, 0, 1\}$ , let us choose an  $\epsilon$ -complex structure  $j^\epsilon \in \mathcal{Q}_{std}$  and let us denote by  $R^\epsilon \subset P$  the stabilizer of  $j^\epsilon$ . (I.e.  $R^\epsilon$  is the group of all para-quaternionic automorphisms of  $(\mathfrak{g}_{-1}, \mathcal{Q}_{std})$  which commute with  $j^\epsilon$ .) Hence each orbit is the homogeneous space  $P/R^\epsilon$  and, by definition, this is the typical fiber of the  $\epsilon$ -twistor bundle  $\mathcal{Z}^\epsilon \rightarrow M$ . Altogether,

$$\mathcal{Z}^\epsilon \cong \mathcal{G} \times_P (P/R^\epsilon) \cong \mathcal{G}/R^\epsilon. \quad (12)$$

Obviously, only the semi-simple part of the left-upper block of  $P$  acts non-trivially on  $\mathcal{Q}_{std}$ , which yields the typical fibers of  $\epsilon$ -twistor bundles are:

- $P/R^- \cong SL(2, \mathbb{R})/SO(2)$ ,
- $P/R^0 \cong SL(2, \mathbb{R})/\mathbb{R}_+$ ,
- $P/R^+ \cong SL(2, \mathbb{R})/SO(1, 1)$ ,

where  $\mathbb{R}_+$  stands for the additive group of real numbers, which is realized as the subgroup  $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$  in  $SL(2, \mathbb{R})$ .

**Remarks and conventions.** (1) The description of the  $\epsilon$ -twistor spaces as in (12) is often taken as a definition. Considering  $\mathcal{G}_0 := \mathcal{G}/P_+$  and  $R_0^\epsilon := G_0 \cap R^\epsilon$ , the same can be written as  $\mathcal{Z}^\epsilon \cong \mathcal{G}_0/R_0^\epsilon$ , cf. [2, 14]. The individual subgroups  $R_0^\epsilon$  are isomorphic to:

- $R_0^- \cong SO(2) \cdot GL(n)$ ,
- $R_0^0 \cong \mathbb{R}_+ \cdot GL(n)$ ,
- $R_0^+ \cong SO(1, 1) \cdot GL(n)$ .

(2) The Cartan geometry  $(\mathcal{G} \rightarrow M, \omega)$  of type  $G/P$  induced by the para-quaternionic structure on  $M$  gives rise to a Cartan geometry  $(\mathcal{G} \rightarrow \mathcal{Z}^\epsilon, \omega)$  of type  $G/R^\epsilon$  on each  $\epsilon$ -twistor space. In particular, the Cartan connection  $\omega$  provides the identification

$$T\mathcal{Z}^\epsilon \cong \mathcal{G} \times_{R^\epsilon} (\mathfrak{g}/\mathfrak{r}^\epsilon),$$

where  $\mathfrak{r}^\epsilon$  is the Lie algebra to  $R^\epsilon$ .

(3) In the concrete computations we preferably use the  $\epsilon$ -complex structures  $j^\epsilon \in \mathcal{Q}_{std}$ , whose  $2 \times 2$ -blocks in the matrix description above are as follows (we do not distinguish in the notation between the discussed block and the whole matrix):

$$j^- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j^0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad j^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (13)$$

An explicit description of the corresponding subgroups  $R^\epsilon$  and their Lie algebras  $\mathfrak{r}^\epsilon$  yields the elements of  $\mathfrak{g}/\mathfrak{r}^\epsilon$  may be represented by the matrices as follows:

$$\left[ \begin{array}{cc|c} b & c & 0 \\ c & -b & 0 \\ \hline X_1 & X_2 & 0 \end{array} \right] \in \mathfrak{g}/\mathfrak{r}^-, \quad \left[ \begin{array}{cc|c} b & 0 & 0 \\ e & -b & 0 \\ \hline X_1 & X_2 & 0 \end{array} \right] \in \mathfrak{g}/\mathfrak{r}^0, \quad \left[ \begin{array}{cc|c} b & d & 0 \\ -d & -b & 0 \\ \hline X_1 & X_2 & 0 \end{array} \right] \in \mathfrak{g}/\mathfrak{r}^+.$$

**4.3. Induced  $\epsilon$ -complex structures.** The current point of view allows an alternative interpretation of the statement we quoted as Proposition 2.3.

**Proposition.** *Let  $p : \mathcal{Z}^\epsilon \rightarrow M$  be the  $\epsilon$ -twistor space,  $\epsilon \in \{-1, 0, 1\}$ , of an almost para-quaternionic manifold  $(M, \mathcal{Q})$ . Then the total space  $\mathcal{Z}^\epsilon$  carries a canonical almost  $\epsilon$ -complex structure  $\mathcal{J}^\epsilon$  such that (6) holds true.*

*Proof.* According to the previous identifications, we search for a canonical  $R^\epsilon$ -invariant  $\epsilon$ -complex structure on  $\mathfrak{g}/\mathfrak{r}^\epsilon$ . For each  $\epsilon \in \{-1, 0, 1\}$ , let  $j^\epsilon$ ,  $R^\epsilon$ , and  $\mathfrak{r}^\epsilon$  be as above. Let us define an endomorphism  $J^\epsilon : \mathfrak{g}/\mathfrak{r}^\epsilon \rightarrow \mathfrak{g}/\mathfrak{r}^\epsilon$  by

$$\begin{bmatrix} U & * \\ X & * \end{bmatrix} \mapsto \begin{bmatrix} Uj^\epsilon & * \\ Xj^\epsilon & * \end{bmatrix} = \begin{bmatrix} U & * \\ X & * \end{bmatrix} \cdot \begin{pmatrix} j^\epsilon & 0 \\ 0 & 0 \end{pmatrix} \pmod{\mathfrak{r}^\epsilon}. \quad (14)$$

Obviously,  $J^\epsilon \circ J^\epsilon = \epsilon \text{id}$ , and it is easy to check that  $J^\epsilon$  is also  $R^\epsilon$ -invariant. Hence it gives rise to an almost  $\epsilon$ -complex structure  $\mathcal{J}^\epsilon$  on  $\mathcal{Z}^\epsilon$ .

By the identification (12), the section  $s : M \rightarrow \mathcal{Z}^\epsilon$  may be represented by a  $P$ -equivariant function  $\sigma : \mathcal{G} \rightarrow P$  such that  $s(x) = u\sigma(u)P$  for any  $u \in \mathcal{G}_x$ . The tangent vector  $\xi \in T_x M$  is represented by the couple  $\llbracket u, X + \mathfrak{p} \rrbracket \in \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ . In these terms, the action of the corresponding  $\epsilon$ -complex structure  $J^s$  on  $T_x M$  is given by

$$\llbracket u, X + \mathfrak{p} \rrbracket \mapsto \llbracket u, X \cdot \text{Ad}_{\sigma(u)} j^\epsilon + \mathfrak{p} \rrbracket, \quad (15)$$

where by  $j^\epsilon$  we again mean the matrix  $\begin{pmatrix} j^\epsilon & 0 \\ 0 & 0 \end{pmatrix}$ . On the other hand, the tangent map to  $T_x s$  is represented by

$$\llbracket u, X + \mathfrak{p} \rrbracket \mapsto \llbracket u\sigma(u), \text{Ad}_{\sigma(u)^{-1}} X + \mathfrak{r}^\epsilon \rrbracket, \quad (16)$$

the  $\epsilon$ -complex structure  $\mathcal{J}^\epsilon$  is determined by (14), and the tangent map to the projection  $p : \mathcal{Z}^\epsilon \rightarrow M$  is clear. Altogether, the composition  $Tp \circ \mathcal{J}^\epsilon \circ Ts$  maps

$$\llbracket u, X + \mathfrak{p} \rrbracket \mapsto \llbracket u\sigma(u), \text{Ad}_{\sigma(u)^{-1}} X \cdot j^\epsilon + \mathfrak{p} \rrbracket. \quad (17)$$

Easily, this indeed coincides with (15), hence  $J^s = Tp \circ \mathcal{J}^\epsilon \circ Ts$ , which was to show.

Moreover, a direct computation shows that if  $J$  is an  $R^\epsilon$ -invariant  $\epsilon$ -complex structure on  $\mathfrak{g}/\mathfrak{r}^\epsilon$  then  $J$  coincides with  $J^\epsilon$  up to the sign (respectively up to a non-zero real multiple) in the case of  $\epsilon = \pm 1$  (respectively  $\epsilon = 0$ ). However, the previous paragraph reveals that the condition (6) is satisfied if and only if  $J$  and  $J^\epsilon$  are equal. Hence the almost  $\epsilon$ -complex structure  $\mathcal{J}^\epsilon$  on  $\mathcal{Z}^\epsilon$  is unique.  $\square$

**Remark.** Note that this way the canonical almost  $\epsilon$ -complex structure on  $\mathcal{Z}^\epsilon$  is determined uniquely and utterly by the underlying almost para-quaternionic structure on  $M$ , independently of any other choice. Of course, this should recover the almost  $\epsilon$ -complex structure that was discussed in 2.3, however this might not be evident at the first glance. Let us briefly explain why this is the case.

Firstly, let us realize the minimal para-quaternionic connections in the sense of [9] are exactly the Weyl connections of the associated parabolic geometry. This follows immediately by definitions: Both the families of connections are connections of the  $G_0$ -structure such that all connections in each family have got the same torsion. In the former case, this is the “minimal” torsion we discussed in remark 2.3(2), in the latter case, it is the harmonic torsion of the associated Cartan connection as we noted in remark 3.2(4). However, as a matter of the construction of the normal Cartan connection, these two torsions do indeed coincide.

Secondly, any such connection  $\nabla$  is given by a reduction  $\mathcal{G}_0 \rightarrow M$  of the Cartan  $P$ -bundle  $\mathcal{G} \rightarrow M$  to  $G_0 \subset P$ . Hence  $TM \cong \mathcal{G}_0 \times_{G_0} (\mathfrak{g}/\mathfrak{p})$  and  $T\mathcal{Z}^\epsilon \cong \mathcal{G}_0 \times_{R_0^\epsilon} (\mathfrak{g}/\mathfrak{r}^\epsilon)$ , where  $R_0^\epsilon = G_0 \cap R^\epsilon$  as before. Under this description, the horizontal subbundle  $H^\nabla \subset T\mathcal{Z}^\epsilon$  is identified with the associated bundle to  $\mathcal{G}_0$  whose standard fiber is  $\mathfrak{g}_-$  (more accurately  $(\mathfrak{g}_- \oplus \mathfrak{r}^\epsilon)/\mathfrak{r}^\epsilon$ ), which is the unique  $R_0^\epsilon$ -invariant subspace in  $\mathfrak{g}/\mathfrak{r}^\epsilon$  which is complementary to  $\mathfrak{p}/\mathfrak{r}^\epsilon$  (which corresponds to the vertical subbundle of the projection  $p : \mathcal{Z}^\epsilon \rightarrow M$ ).

Now, the original description of  $\mathcal{J}^\epsilon$  in terms of a representative affine connection, which we roughly recalled in remark 2.3(1), can be snugly compared with the current invariant approach.

**4.4. Decomposition of forms.** The decomposition of complex forms into  $(p, q)$ -types has got the following natural generalization. Suppose we are given a real vector space  $W$  endowed with an endomorphism  $A \in \text{End}(W)$  such that  $A^2 = A \circ A$  is a multiple of the identity, written  $A^2 = -|A|^2 \text{id}$  as in (5). Below we consider especially  $W$ -valued bilinear maps  $\varphi : W \times W \rightarrow W$ . The notion of  $(p, q)$ -type of  $\varphi$  with respect to  $A$  is characterized as follows:

- type  $(1, 1)$ :  $\varphi(AX, AY) = -A^2\varphi(X, Y) = |A|^2\varphi(X, Y)$ ,
- type  $(0, 2)$ :  $\varphi(AX, Y) = \varphi(X, AY) = -A\varphi(X, Y)$ ,
- type  $(2, 0)$ :  $\varphi(AX, Y) = \varphi(X, AY) = A\varphi(X, Y)$ ,

where  $X$  and  $Y$  are arbitrary vectors from  $W$ . The space of maps of type  $(p, q)$  with respect to  $A$  will be denoted by  $\bigotimes_A^{p,q} W^* \otimes W$ .

If  $|A|^2 \neq 0$  then  $\varphi$  decomposes uniquely into the sum of components of particular types with respect to  $A$ , namely  $\varphi = \varphi_A^{2,0} + \varphi_A^{1,1} + \varphi_A^{0,2}$ , where

$$\begin{aligned} \varphi_A^{1,1}(X, Y) &= \frac{1}{2|A|^2} (|A|^2\varphi(X, Y) + \varphi(AX, AY)), \\ \varphi_A^{0,2}(X, Y) &= \frac{1}{4|A|^2} (|A|^2\varphi(X, Y) - \varphi(AX, AY) + A\varphi(AX, Y) + A\varphi(X, AY)), \\ \varphi_A^{2,0}(X, Y) &= \frac{1}{4|A|^2} (|A|^2\varphi(X, Y) - \varphi(AX, AY) - A\varphi(AX, Y) - A\varphi(X, AY)). \end{aligned} \tag{18}$$

These facts can be formulated so that  $\bigotimes^2 W^* \otimes W$  is a direct sum of the three particular subspaces  $\bigotimes_A^{p,q} W^* \otimes W$  and the prescription  $\varphi \mapsto \varphi_A^{p,q}$  defines a projection  $\bigotimes^2 W^* \otimes W \rightarrow \bigotimes_A^{p,q} W^* \otimes W$ .

If  $|A|^2 = 0$ , the notion of type of  $\varphi$  is of course rather degenerate. E.g., there are forms which are simultaneously of all three types with respect to  $A$ , hence any type decomposition is a priori meaningless. However it might seem strange, let us consider the map  $\bigotimes^2 W^* \otimes W \rightarrow \bigotimes_A^{0,2} W^* \otimes W$ ,  $\varphi \mapsto \varphi_A^{0,2}$ , given by

$$\varphi_A^{0,2}(X, Y) := \frac{1}{4} (-\varphi(AX, AY) + A\varphi(AX, Y) + A\varphi(X, AY)). \tag{19}$$

**Convention.** Let  $\varphi \in \bigotimes^2 W^* \otimes W$  and  $A \in \text{End}(W)$  such that  $A^2 = -|A|^2 \text{id}$ . By the  $(0, 2)$ -part of  $\varphi$  with respect to  $A$  we always mean the element  $\varphi_A^{0,2}$ , which is defined either by (18) if  $|A|^2 \neq 0$  or by (19) if  $|A|^2 = 0$ .

If  $W$  carries a para-quaternionic structure  $\mathcal{Q}_W \subset \text{End}(W)$ , we say that  $\varphi$  is of type  $(p, q)$  with respect to  $\mathcal{Q}_W$  if it is of type  $(p, q)$  with respect to any  $A \in \mathcal{Q}_W$ . The space of maps of type  $(p, q)$  with respect to  $\mathcal{Q}_W$  is simply denoted by  $\bigotimes^{p,q} W^* \otimes W$ .

The notion of type  $(1, 1)$  has got clear meaning also for real-valued bilinear maps, the corresponding subspace in  $\otimes^2 W^*$  is denoted by  $\otimes^{1,1} W^*$ . There is a natural projection  $\otimes^2 W^* \rightarrow \otimes^{1,1} W^*$ ,  $\varphi \mapsto \varphi^{1,1}$ , given by

$$\varphi^{1,1}(X, Y) := \frac{1}{4} (\varphi(X, Y) - \varphi(IX, IY) - \varphi(JX, JY) + \varphi(KX, KY)), \quad (20)$$

where  $(I, J, K)$  is an arbitrary basis of  $\mathcal{Q}_W$ , i.e. a triple satisfying (4). The kernel of this projection is a natural complementary subspace to  $\otimes^{1,1} W^*$  in  $\otimes^2 W^*$ . In particular, this yields the decomposition

$$\Lambda^2 W^* = \Lambda^{1,1} W^* \oplus \ker \pi^{1,1}, \quad (21)$$

where  $\pi^{1,1}$  denotes the restriction of the projection above to  $\Lambda^2 W^* \subset \otimes^2 W^*$ .

From 4.1 we know the para-quaternionic structure on a  $2n$ -dimensional vector space  $W$  is equivalent to the identification  $W \cong U^* \otimes V$ , where  $U$  and  $V$  are vector spaces of dimensions 2 and  $n$ , respectively. This identification yields the decomposition

$$\Lambda^2 W^* = (\Lambda^2 U \otimes S^2 V^*) \oplus (S^2 U \otimes \Lambda^2 V^*). \quad (22)$$

It turns out this decomposition agrees with the one in (21):

**Lemma.** *With the previous notation, the following equalities hold:*

$$\Lambda^2 U \otimes S^2 V^* = \Lambda^{1,1}(U \otimes V^*) \text{ and } S^2 U \otimes \Lambda^2 V^* = \ker \pi_{1,1}.$$

*Proof.* It is a straightforward exercise to show the left inclusions hold in both cases. Now, the equalities follow from the complementarity of the respective subspaces on right- and left-hand sides, i.e. from (21) and (22).  $\square$

**4.5. Integrability revised.** Under the general setup from 4.2, the torsion of the Cartan geometry  $(\mathcal{G} \rightarrow M, \omega)$  and  $(\mathcal{G} \rightarrow \mathcal{Z}^\epsilon, \omega)$  is denoted by  $\tau \in \Omega^2(M, TM)$  and  $\mathcal{T} \in \Omega^2(\mathcal{Z}^\epsilon, T\mathcal{Z}^\epsilon)$ , respectively. By definitions,  $\mathcal{T}$  is strictly horizontal with respect to the projection  $p : \mathcal{Z}^\epsilon \rightarrow M$ , hence  $\tau(\xi, \eta) = Tp(\mathcal{T}(\hat{\xi}, \hat{\eta}))$ , where  $\hat{\xi}, \hat{\eta} \in T\mathcal{Z}^\epsilon$  are any lifts of  $\xi, \eta \in TM$ . In other words,

$$\tau = Tp \circ \mathcal{T} \circ (Ts \times Ts) \quad (23)$$

for any section  $s : M \rightarrow \mathcal{Z}^\epsilon$ .

The following lemma can be seen as a Cartan-geometric analog of the well-known fact that an almost (para-)complex structure is integrable if and only if the  $(0, 2)$ -part of the torsion of some (and consequently any) compatible affine connection vanishes. The reasoning below is very similar to the one in subsection 4.4.10 in [7], which allows us some abbreviations. Note also that an alternative treatment in the case  $\epsilon = 1$  can be found in section 5 in [2].

**Lemma.** *Let  $\mathcal{Z}^\epsilon$  be the  $\epsilon$ -twistor space with the canonical almost  $\epsilon$ -complex structure,  $\epsilon \in \{-1, 0, 1\}$ , and let  $\mathcal{T}$  be the torsion of the associated Cartan connection over  $\mathcal{Z}^\epsilon$ . Then the Nijenhuis tensor of  $\mathcal{J}^\epsilon$  is a non-zero constant multiple of the  $(0, 2)$ -part of  $\mathcal{T}$  with respect to  $\mathcal{J}^\epsilon$ , according to the convention 4.4.*

*Proof.* To deal efficiently with the tensor fields on  $\mathcal{Z}^\epsilon$  we use the corresponding frame forms with respect to the Cartan connection  $\omega$ . On the one hand, the frame form of the torsion  $\mathcal{T}$  is the  $R^\epsilon$ -equivariant functions  $\mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{r}^\epsilon)^* \otimes (\mathfrak{g}/\mathfrak{r}^\epsilon)$ , which assigns to each  $u \in \mathcal{G}$  the bilinear map

$$(X + \mathfrak{r}^\epsilon, Y + \mathfrak{r}^\epsilon) \mapsto \pi([X, Y] - \omega([\omega^{-1}(X)(u), \omega^{-1}(Y)(u)])), \quad (24)$$

where  $\pi$  is the projection  $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{r}^\epsilon$  as before. Similarly, the frame form of  $\mathcal{J}^\epsilon$  is the constant function  $\mathcal{G} \rightarrow (\mathfrak{g}/\mathfrak{r}^\epsilon)^* \otimes (\mathfrak{g}/\mathfrak{r}^\epsilon)$  with the value  $J^\epsilon$ , which is described in (14). Now one can express the frame form of  $\mathcal{T}_{\mathcal{J}^\epsilon}^{0,2}$  following the conventions from 4.4, distinguishing the cases  $\epsilon \neq 0$  and  $\epsilon = 0$ .

On the other hand, the frame form of the Nijenhuis tensor  $N_{\mathcal{J}^\epsilon}$  is the equivariant function, which assigns to each  $u \in \mathcal{G}$  the bilinear map

$$\begin{aligned} (X + \mathfrak{r}^\epsilon, Y + \mathfrak{r}^\epsilon) \mapsto & -(J^\epsilon)^2(\pi(\omega([\omega^{-1}(X)(u), \omega^{-1}(Y)(u)]))) - \\ & -\pi(\omega([\omega^{-1}(J^\epsilon X)(u), \omega^{-1}(J^\epsilon Y)(u)])) + \\ & + J^\epsilon(\pi(\omega([\omega^{-1}(J^\epsilon X)(u), \omega^{-1}(Y)(u)]))) + \\ & + J^\epsilon(\pi(\omega([\omega^{-1}(X)(u), \omega^{-1}(J^\epsilon Y)(u)]))), \end{aligned} \quad (25)$$

where  $J^\epsilon X$  denotes any element in  $\mathfrak{g}$  such that  $\pi(J^\epsilon X) = J^\epsilon(\pi(X))$ . (Note that by the  $R^\epsilon$ -invariance of the  $\epsilon$ -complex structure  $J^\epsilon$ , this is indeed a well-defined object.)

Now let us consider the tensor field  $\mathcal{S} := N_{\mathcal{J}^\epsilon} - 4\mathcal{T}_{\mathcal{J}^\epsilon}^{0,2}$ . Just a simple substituting, taking into accounts that  $(J^\epsilon)^2 = \epsilon \text{id}$ , shows the frame form of  $\mathcal{S}$  is the constant function assigning to each  $u \in \mathcal{G}$  the bilinear map

$$\begin{aligned} (X + \mathfrak{r}^\epsilon, Y + \mathfrak{r}^\epsilon) \mapsto & -(J^\epsilon)^2(\pi([X, Y])) - \pi([J^\epsilon X, J^\epsilon Y]) + \\ & + J^\epsilon(\pi([J^\epsilon X, Y])) + J^\epsilon(\pi([X, J^\epsilon Y])). \end{aligned} \quad (26)$$

However, by the definition of  $J^\epsilon$  in (14) it immediately follows that

$$(J^\epsilon)^2(\pi([X, Y])) = J^\epsilon(\pi([X, J^\epsilon Y])) \text{ and } \pi([J^\epsilon X, J^\epsilon Y]) = J^\epsilon(\pi([J^\epsilon X, Y])),$$

for any  $\epsilon \in \{-1, 0, 1\}$ . Therefore  $\mathcal{S} = 0$ , which completes the proof.  $\square$

Here is the promised extension and reinterpretation of the statement we cited as Theorem 2.4. An analog of this statement in the four-dimensional case is formulated in 6.1.

**Proposition.** *Let  $(M, \mathcal{Q})$  be an almost para-quaternionic manifold of dimension  $2n > 4$ . Let  $(\mathcal{Z}^\epsilon, \mathcal{J}^\epsilon)$  be the  $\epsilon$ -twistor space with the canonical almost  $\epsilon$ -complex structure, where  $\epsilon \in \{-1, 0, 1\}$ . Then  $\mathcal{Q}$  is integrable if and only if  $\mathcal{J}^\epsilon$  is integrable.*

*Proof.* Seeing the almost para-quaternionic structure as an almost Grassmannian structure, we know from Proposition 3.2 and the subsequent remarks that its integrability is equivalent to the vanishing of the torsion  $\tau$ , which equals to the harmonic torsion  $T$ , of the associated normal Cartan connection  $\omega$  over  $M$ . By the previous lemma, the integrability of  $\mathcal{J}^\epsilon$  is equivalent to the vanishing of the  $(0, 2)$ -part (with respect to  $\mathcal{J}^\epsilon$ ) of the torsion  $\mathcal{T}$  of  $\omega$  understood as a Cartan connection over  $\mathcal{Z}^\epsilon$ .

Let  $\mathcal{Q}$  be integrable, i.e.  $\tau = T = 0$ . Hence the whole Cartan curvature is determined by the harmonic component  $K$  of homogeneity two. By the description in remark 3.2(2), the frame form of  $K$  takes values in  $\mathfrak{sl}(n, \mathbb{R})$ , the lower right block in  $\mathfrak{g}_0$ , and the curvature component of homogeneity three has necessarily values in  $\mathfrak{p}_+$ . Hence, for any  $\epsilon$ , the Cartan curvature takes values in  $\mathfrak{r}^\epsilon$ . This means the torsion  $\mathcal{T}$  is also trivial, i.e.  $\mathcal{J}^\epsilon$  is integrable.

Conversely, let  $\mathcal{J}^\epsilon$  be integrable, i.e. the torsion  $\mathcal{T}$  has got trivial  $(0, 2)$ -part with respect to  $\mathcal{J}^\epsilon$ . The main consequence of the relations (6) and (23) is that  $\tau$  must have vanishing  $(0, 2)$ -part with respect to any  $\epsilon$ -complex structure contained in  $\mathcal{Q}$ . Since  $\tau$  coincides with the harmonic torsion  $T$ , its values are heavily restricted as

explained in remark 3.2(2). Namely, the frame form of  $\tau = T$  takes values in the irreducible  $G_0$ -representation  $\mathbb{E} := \ker \square \cap \Lambda^2 \mathfrak{g}_1 \otimes \mathfrak{g}_{-1}$ . For each  $\epsilon$ , we are going to show that there is actually no non-zero element of  $\mathbb{E}$  which would satisfy the requirement.

In our case,  $\mathbb{E}$  is characterized as the intersection of kernels of the two obvious traces in

$$S^2 \mathbb{R}^2 \otimes \mathbb{R}^{2*} \otimes \Lambda^2 \mathbb{R}^{n*} \otimes \mathbb{R}^n \subset \Lambda^2 \mathfrak{g}_1 \otimes \mathfrak{g}_{-1}.$$

Let  $(e_i)$  be the standard basis of  $\mathbb{R}^2$  and  $(\varepsilon^i)$  the dual basis of  $\mathbb{R}^{2*}$ . Let  $(u_i)$  be the standard basis of  $\mathbb{R}^n$  and  $(v^i)$  the dual basis of  $\mathbb{R}^{n*}$ . Finally, let us consider

$$\varphi := e_1 \odot e_1 \otimes \varepsilon^2 \otimes v^1 \wedge v^2 \otimes u_3.$$

Evidently,  $\varphi \in \mathbb{E}$  and it easily follows  $\varphi$  is of type  $(0, 2)$  with respect to the 1-complex structure  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . Similarly, it is a short exercise to show that  $\varphi$  has

got non-trivial  $(0, 2)$ -part with respect to the  $(-1)$ -complex structure  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

and also with respect to the 0-complex structure  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ . Hence, for any  $\epsilon$ , it is not true that the element  $\varphi \in \mathbb{E}$  has got vanishing  $(0, 2)$ -part with respect to all  $\epsilon$ -complex structures from  $\mathcal{Q}_{std}$ . Since  $\mathbb{E}$  is an irreducible representation, there is no non-zero element in  $\mathbb{E}$  with vanishing  $(0, 2)$ -part with respect to all  $\epsilon$ -complex structures from  $\mathcal{Q}_{std}$ .

Altogether, the requirement on  $\tau$  is satisfied if and only if  $\tau = 0$ , which is equivalent to the integrability of  $\mathcal{Q}$ .  $\square$

**4.6. The role of 0-twistor space.** During the material we have met number of fiber bundles over  $M$ . Some of them are somehow related — let us remind several identifications in chronological order:

In subsection 2.3, the bundle  $\mathcal{Q} \subset \text{End}(TM)$  defining an almost para-quaternionic structure on  $M$  is decomposed into disjoint subbundles such that  $\mathcal{Q}^0 \subset \mathcal{Q}$  coincides with the 0-twistor space  $\mathcal{Z}^0$ . For the corresponding almost Grassmannian structure  $E^* \otimes F \cong TM$ , the projectivization of the auxiliary vector bundle  $E$  is identified with the bundle  $\mathcal{E}$  of  $\beta$ -planes in  $TM$ , see remark 3.1(3). This bundle is further identified with the correspondence space  $\mathcal{CM}$  by Proposition 3.3(1).

The following statement supplies another identification to the collection above. This is the promised link between the two notions of twistor correspondence we mentioned for almost para-quaternionic and almost Grassmannian structures, respectively. The statement is by no means surprising, but presumably one could not say it is evident a priori:

**Proposition.** *Let  $\mathcal{Z}^0$  be the 0-twistor space over an almost para-quaternionic manifold  $M$ , let  $\mathcal{CM}$  be the correspondence space of the equivalent almost Grassmannian structure on  $M$ , and let  $D \subset TCM$  be the vertical subbundle of the projection  $\mathcal{CM} \rightarrow M$ .*

*Then  $\mathcal{Z}^0$  is naturally identified with  $D$ . In particular,  $\mathcal{P}\mathcal{Z}^0$  is identified with  $\mathcal{CM}$ .*

*Proof.* As an associated bundle over  $\mathcal{CM}$ , the line bundle  $D$  is identified with  $\mathcal{G} \times_Q (\mathfrak{p}/\mathfrak{q})$ , see (9). The action of  $Q$  on  $\mathfrak{p}/\mathfrak{q}$  is transitive and the stabilizer of any element is just the subgroup  $R^0$  from 4.2. Hence  $D \cong \mathcal{G} \times_Q (Q/R^0) \cong \mathcal{G}/R^0 \cong \mathcal{Z}^0$ , according to (12).  $\square$

## 5. Remarks on the integrable case

In the integrable case, several interesting phenomena appears. Firstly, the induced geometric structure on the correspondence space  $\mathcal{CM}$  turns out to be locally equivalent to a (torsion-free) path geometry on a leaf space  $X$  in the sense of [10]. This allows yet another interpretation of the 0-twistor space. Secondly, there are special classes of compatible affine connections on  $M$ , whose holonomy groups are reduced to  $R_0^5 \subset G_0$ . We briefly mention an easy criterion in terms of specific scales.

**5.1. Torsion-free path geometries and the 0-twistor space again.** As a consequence of statement (2) in Proposition 3.3, an integrable Grassmannian structure on  $M$  gives rise to a generalized path geometry on  $\mathcal{CM}$ . The one-dimensional distribution  $D \subset T\mathcal{CM}$  is just the vertical subbundle of the projection  $\mathcal{CM} \rightarrow M$ . If  $n > 2$ , the distribution  $V \subset T\mathcal{CM}$  is automatically involutive, which allows to construct a local leaf space  $X$ . Proposition 4.4.4. in [7] then shows that  $\mathcal{CM}$  is locally identified with  $\mathcal{PTX}$  such that the subbundle  $V$  corresponds to the vertical subbundle of  $\mathcal{PTX} \rightarrow X$ . In particular, the generalized path geometry on  $\mathcal{CM}$  is locally equivalent to a path geometry on  $X$  so that the points in  $M$  corresponds to the paths in  $X$ . (By a path geometry on  $X$  we mean a system of unparameterized curves on  $X$ , which are determined by a tangent direction in one point. Path geometries are typically given by systems of ODE's of second order.) According to statement (3) in Proposition 3.3, the path geometry is called torsion-free, which perfectly agrees with the terminology of [10].

Let us advert to some details in the proof of Proposition 4.4.4 in [7]: Denoting by  $\psi : \mathcal{CM} \supset U \rightarrow X$  the local leaf space projection, its tangent map  $T\psi$  induces a linear isomorphism  $T_x U/V_x \rightarrow T_{\psi(x)} X$ , for any  $x \in U$ . Hence  $D_x \subset T_x U$  projects to a one-dimensional subspace in  $T_{\psi(x)} X$ , i.e. an element in  $\mathcal{PT}_{\psi(x)} X$  which is denoted as  $\tilde{\psi}(x)$ . It is then shown the tangent map to  $\tilde{\psi} : U \rightarrow \mathcal{PTX}$  is invertible, therefore  $\tilde{\psi}$  is an open embedding.

It is easy to see that  $\tilde{\psi}$  extends to a local embedding of  $D$  into  $TX$ . Hence, in addition to all the previous identifications, we claim:

**Proposition.** *Let  $\mathcal{Z}^0$  be the 0-twistor space over an integrable para-quaternionic manifold  $M$ , let  $\mathcal{CM}$  be the correspondence space of the equivalent Grassmannian structure on  $M$ , and let  $X$  be a local leaf space of the foliation determined by the involutive distribution  $V \subset T\mathcal{CM}$ .*

*Then  $\mathcal{Z}^0$  is locally identified with the tangent bundle  $TX$  such that the rank  $n+1$  distribution  $\ker \mathcal{J}^0 \subset T\mathcal{Z}^0$  corresponds to the vertical subbundle of the canonical projection  $TX \rightarrow X$ .*

*Proof.* From Proposition 4.6 we know that  $\mathcal{Z}^0 \cong D$  and, by the previous counting,  $D$  is locally identified with  $TX$ . Hence  $\mathcal{Z}^0$  is locally identified with  $TX$ .

The projection  $\mathcal{Z}^0 \cong TX \rightarrow X$  factorizes through  $\mathcal{PTX} \cong \mathcal{CM}$  and we already know the vertical subbundle of  $\mathcal{PTX} \rightarrow X$  coincides with  $V$ . It is enough to show that, under the canonical projection  $\mathcal{Z}^0 \rightarrow \mathcal{PZ}^0$ ,  $\ker \mathcal{J}^0$  maps to  $V$ : By the proof of Proposition 4.3, the almost complex structure  $\mathcal{J}^0$  corresponds to the  $R^0$ -invariant endomorphism  $J^0 : \mathfrak{g}/\mathfrak{r}^0 \rightarrow \mathfrak{g}/\mathfrak{r}^0$  given by (14). With the same conventions as before, the kernel of  $J^0$  is the  $R^0$ -invariant subspace represented by the matrices of

the form

$$\left[ \begin{array}{cc|c} u & 0 & 0 \\ 0 & -u & 0 \\ 0 & X_2 & 0 \end{array} \right].$$

The tangent map to the canonical projection  $\mathcal{Z}^0 \cong \mathcal{G}/R^0 \rightarrow \mathcal{G}/Q \cong \mathcal{P}\mathcal{Z}^0$  corresponds to the obvious  $R^0$ -invariant projection  $\mathfrak{g}/\mathfrak{r}^0 \rightarrow \mathfrak{g}/\mathfrak{q}$  determined by  $\mathfrak{r}^0 \subset \mathfrak{q}$ . The image of  $\ker J^0$  in  $\mathfrak{g}/\mathfrak{q}$  is then represented by

$$\left[ \begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & X_2 & 0 \end{array} \right].$$

Now we see the image coincides with the  $Q$ -invariant subspace  $\mathfrak{p}'/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ , which defines the distribution  $V \subset TCM$  as in (9).  $\square$

**5.2. Specific scales and exotic holonomies.** Let  $\mathcal{Z}^\epsilon$  be the  $\epsilon$ -twistor space over an integrable para-quaternionic manifold  $M$ . Sections of the projection  $\mathcal{Z}^\epsilon \rightarrow M$  are by definition (almost)  $\epsilon$ -complex structures on  $M$ . The structure corresponding to a section  $s : M \rightarrow \mathcal{Z}^\epsilon$  is denoted, in accordance with (6), by  $J^s$ .

Since  $\mathcal{Z}^\epsilon \cong \mathcal{G}/R^\epsilon$ , the considered sections are further in a bijective correspondence with reductions of the principal bundle  $\mathcal{G} \rightarrow M$  to the structure group  $R^\epsilon \subset P$ . Equivalently,  $\mathcal{Z}^\epsilon \cong \mathcal{G}_0/R_0^\epsilon$  and the sections of this bundle are in a bijective correspondence with reductions of the bundle  $\mathcal{G}_0 \rightarrow M$  to  $R_0^\epsilon \subset G_0$ . Since  $R_0^\epsilon$  is actually contained in  $G_0^{ss}$ , the semi-simple part of  $G_0$ , the projection  $\mathcal{Z}^\epsilon \rightarrow M$  factorizes through the canonical projection  $\mathcal{Z}^\epsilon \cong \mathcal{G}_0/R_0^\epsilon \rightarrow \mathcal{G}_0/G_0^{ss}$ . The latter space, denoted by  $\mathcal{L}$ , is a scale bundle of the Grassmannian structure on  $M$ , see A.3(2). (Note that in our case the center of  $\mathfrak{g}_0$  is 1-dimensional, hence any scaling element is a multiple of the grading element of  $\mathfrak{g}$ ). Hence any section of  $\mathcal{Z}^\epsilon \rightarrow M$  also gives rise to a section of the scale bundle  $\mathcal{L} \rightarrow M$ . In particular, any section  $s : M \rightarrow \mathcal{Z}^\epsilon$  determines an exact Weyl connection on  $M$ , which is denoted by  $\nabla^s$ .

By the general principles mentioned in A.3 and in remark 3.2(4), any Weyl connection of an integrable Grassmannian structure is torsion-free with the holonomy group contained in  $G_0$ , whereas holonomy groups of exact Weyl connections are contained in  $G_0^{ss} \subset G_0$ . There is an easy characterization of those sections  $s$  for which the corresponding (almost)  $\epsilon$ -complex structure  $J^s$  is parallel with respect to  $\nabla^s$ . If this is the case, the holonomy group is further reduced:

**Proposition.** *Let  $\mathcal{Z}^\epsilon \rightarrow M$  be the  $\epsilon$ -twistor space over an integrable para-quaternionic manifold  $M$ , let  $s : M \rightarrow \mathcal{Z}^\epsilon$  be a section of the previous projection, and let  $J^s$  and  $\nabla^s$  be the corresponding (almost)  $\epsilon$ -complex structure and exact Weyl connection on  $M$ , respectively.*

*Then  $\nabla^s J^s = 0$  if and only if  $s(M) \subset \mathcal{Z}^\epsilon$  is an  $\epsilon$ -complex submanifold. In that case, the holonomy group of  $\nabla^s$  is a subgroup in  $R_0^\epsilon \subset G_0^{ss}$ .*

*Proof.* In remark 4.3 we have described the unique  $R_0^\epsilon$ -invariant complementary subspace to  $\mathfrak{p}/\mathfrak{r}^\epsilon$  in  $\mathfrak{g}/\mathfrak{r}^\epsilon$ . The latter subspace corresponds to the vertical subspace of the projection  $\mathcal{Z}^\epsilon \rightarrow M$ , while the complement corresponds to the horizontal distribution  $H^{\nabla^s} \subset T\mathcal{Z}^\epsilon$  of the induced Weyl connection  $\nabla^s$ .

Now, it turns out this subspace is also the unique complement which is invariant with respect to the  $\epsilon$ -complex structure  $J^\epsilon$  on  $\mathfrak{g}/\mathfrak{r}^\epsilon$ . The geometric interpretation of this observation is that  $s(M) \subset \mathcal{Z}^\epsilon$  is an  $\epsilon$ -complex submanifold if and only if

the image of the tangent map  $T_x s : T_x M \rightarrow T_{s(x)} \mathcal{Z}^\epsilon$  coincides with  $H_{s(x)}^{\nabla^s}$  for any  $x \in M$ . The latter condition is clearly equivalent to  $\nabla^s J^s = 0$ .

Since  $R^\epsilon$  is exactly the subgroup of automorphisms of  $\mathfrak{g}/\mathfrak{r}^\epsilon$  commuting with  $J^\epsilon$  and  $R_0^\epsilon = G_0 \cap R^\epsilon$ , the rest follows immediately.  $\square$

**Remark.** A careful study in the most interesting case  $\epsilon = -1$  can be found in [14]. In that case it is  $R_0^- \cong S^1 \cdot GL(n, \mathbb{R})$ . Note that this group appears as an exotic (irreducible) holonomy group of a torsion-free affine connection with quite an interesting history, see remark 8 in [6]. The cases corresponding to  $\epsilon = 0$  and 1 obviously lead to reducible holonomy groups.

## 6. Final remarks

**6.1. Dimension four.** We have mentioned several times the case that the base manifold  $M$  has got dimension four is quite specific. Let us briefly specify this specificity. According to the previous notation, this corresponds to  $n = 2$ , i.e. we now think of almost Grassmannian structures of type  $(2, 2)$ . It is well known that almost Grassmannian structures of this type (hence also almost para-quaternionic structures in dimension 4) are equivalent to conformal structures of the split signature. This fact easily follows from the isomorphism of the respective structure groups, see below. In terms of distinguished directions in the tangent bundle  $TM$ , the equivalence may be stated so that the Segre cone of  $TM \cong E^* \otimes F$  is just the cone of the non-zero null-vectors of the conformal structure or vice versa. (Note that the Segre cone forms a hyper-quadric in the tangent space exactly in this dimension).

On the Lie algebra level, the block corresponding to  $\mathfrak{g}_{-1}$  (in the block description from 3.2) has got the size  $2 \times 2$ . Let us consider the quadratic form on  $\mathfrak{g}_{-1}$  defined by the determinant; the corresponding polar form is denoted by  $\delta$  for later purposes. Evidently, the null-vectors of this form exhaust exactly the Segre cone of rank-one elements in  $\mathfrak{g}_{-1} \cong \mathbb{R}^{2*} \otimes \mathbb{R}^2$ . The adjoint action of  $G_0$  on  $\mathfrak{g}_{-1}$  changes the form conformally, which leads to the identification  $G_0 \cong CSO_0(2, 2)$ . For oriented almost Grassmannian structures, the structure group is a two-fold covering of the just mentioned one and the previous reasoning leads to an isomorphism  $G_0 \cong CSpin(2, 2)$ . Under this identification, the bundles  $E$  and  $F$  are identified with the two spinor bundles. Hence, by Proposition 3.3(1), the correspondence space  $\mathcal{CM}$  is identified with the projectivized spinor bundle. Anyway, the harmonic curvature of Grassmannian structures of the considered type has got two components of homogeneity two, which corresponds to the self-dual and the anti-self-dual part of the conformal Weyl curvature. For more details, see e.g. subsection 4.1.4 in [7].

Now, it is an easy observation that  $\delta$  is, up to a non-zero constant multiple, the unique non-degenerate bilinear form on  $\mathfrak{g}_{-1}$  which is of type  $(1, 1)$  with respect to the standard para-quaternionic structure  $\mathcal{Q}_{std}$ . This means that, for any  $A \in \mathcal{Q}_{std}$  and  $X, Y \in \mathfrak{g}_{-1}$ , the following holds:

$$\delta(AX, AY) = |A|^2 \delta(X, Y).$$

Note that, if  $|A|^2 \neq 0$  then this condition is equivalent to

$$\delta(AX, Y) + \delta(X, AY) = 0,$$

i.e.  $A$  is skew with respect to  $\delta$ ; for  $|A|^2 = 0$ , the latter condition is stronger. Conversely, it turns out that if  $A$  is an endomorphism of  $\mathfrak{g}_{-1}$ , which is skew with respect to  $\delta$  and whose square  $A^2$  is a multiple of the identity, then  $A$  belongs to

$\mathcal{Q}_{std}$ . Altogether, we have got a characterization of the standard para-quaternionic structure in terms of  $\delta$ , which is obviously independent on a multiple of  $\delta$ . The geometric interpretation of these observations is the following: An endomorphism  $A$  of the tangent bundle of an almost para-quaternionic 4-manifold  $(M, \mathcal{Q})$  belongs to  $\mathcal{Q} \subset \text{End}(TM)$  if and only if  $A \circ A$  is a multiple of the identity and  $A$  is skew with respect to any metric form the conformal class of the corresponding conformal structure.

Let us now consider a  $\epsilon$ -twistor space with the canonical almost  $\epsilon$ -complex structure as in 4.3. The characterization of its integrability discussed in 4.5 can be dealt very similarly also in this dimension: Lemma 4.5 holds true in general, but Proposition 4.5 evidently needs an intervention. In the current case, the torsion of the normal Cartan connection automatically vanishes, hence this is no relevant condition. According to the description of the harmonic curvatures in remark 3.2(2), it should be obvious that vanishing of the component  $K_1$  is a sufficient condition for the integrability of the induced almost  $\epsilon$ -complex structure. The fact that this condition is also necessary follows by the same scenario as in the proof of Proposition 4.5.

**Proposition.** *Let  $(M, \mathcal{Q})$  be a 4-dimensional almost para-quaternionic manifold and let  $[g]$  be the corresponding conformal structure on  $M$ .*

- (1) *The  $\epsilon$ -twistor space  $\mathcal{Z}^\epsilon$  is identified with the subbundle of  $\text{End}(TM)$  of those elements which square to  $\epsilon \text{id}$  and which are skew with respect to  $[g]$ .*
- (2) *The canonical almost  $\epsilon$ -complex structure on  $\mathcal{Z}^\epsilon$  is integrable if and only if  $(M, [g])$  is anti-self-dual (or self-dual according to the conventions).*

For  $\epsilon = -1$ , the characterization of the respective twistor space may be shortened by saying that  $\mathcal{Z}^{-1}$  consists of orthogonal (almost) complex structures in  $TM$ . This should resemble the classical formulations.

**6.2. Compatible metrics.** It is a very important situation if there exists a (pseudo-)Riemannian metric which is compatible with the given geometric structure. This is a rather strong condition which is, together with its consequences, thoroughly studied both from the para-quaternionic and the Grassmannian point of view. Following [2, 3, 4], let us quickly summarize some of the classical issues here.

First of all, let us suggest a natural decomposition of the bundle  $S^2 T^* M$  in the spirit of 4.4. The almost para-quaternionic structure  $\mathcal{Q} \subset \text{End}(TM)$  induces the decomposition

$$S^2 T^* M = S^{1,1} T^* M \oplus \ker \pi_{1,1},$$

where  $\pi_{1,1} : S^2 T^* M \rightarrow S^{1,1} T^* M$  is the restriction of the natural projection (20) to  $S^2 T^* M$ . On the other hand, the corresponding almost Grassmannian structure  $TM \cong E^* \otimes F$  yields

$$S^2 T^* M = (\Lambda^2 E \otimes \Lambda^2 F^*) \oplus (S^2 E \otimes S^2 F^*).$$

As in lemma 4.4, the two decompositions perfectly agree so that

$$\Lambda^2 E \otimes \Lambda^2 F^* \cong S^{1,1}(E^* \otimes F) \text{ and } S^2 E \otimes S^2 F^* \cong \ker \pi_{1,1}.$$

Now, the metric on  $M$  is *compatible* with the given geometric structure if it is a section just of the first summand in  $S^2 T^* M$  according to the decomposition above. In order for the metric to be non-degenerate, the rank of the vector bundle  $F$  has to be even. Hence, if there is a compatible metric then the dimension of the base

manifold is divisible by 4, say  $\dim M = 4k$ . It is also obvious, that all tangent vectors in the Segre cone are null with respect to any compatible metric. Since the Segre cone contains linear subspaces of maximal dimension  $n = 2k$ , the compatible metric is necessarily of split signature  $(2k, 2k)$ .

Much more restrictive situation is as follows. If there is a compatible metric on  $M$  and the Levi-Civita connection of this metric is a compatible connection of the geometric structure, then the metric (and the geometric structure as well) is called *para-quaternionic Kähler*. Since Levi-Civita connection is torsion free, para-quaternionic Kähler structures are automatically integrable. It is also the case, that para-quaternionic Kähler metric is necessarily Einstein. According to the interpretations we recalled in 6.1, the feature of the structure to be para-quaternionic Kähler descends in the 4-dimensional case to the property that the corresponding conformal structure is (anti-)self-dual and contains an Einstein metric in the conformal class.

Note that in [4], very similar problem is dealt in holomorphic category with some minor additional assumptions and slightly different vocabulary. In particular, there is a characterization of the existence of such a metric in terms of solutions of an invariant overdetermined system of PDE's on  $M$ . It turns out, these results may be adapted to our real form so that the solutions of the mentioned system are in a bijective correspondence with sections of an appropriate tractor bundle which are in the kernel of the first BGG operator; see [11] for the terminology and different examples of similar character.

**6.3. Outlook.** Besides the topics discussed so far, there are another ideas around we have not even mentioned. Let us formulate at least some of them.

(1) In subsection 5.1 we have quickly recalled the notion of torsion-free path geometries with a reference to [10]. Note, however, the approach in there is generally opposite to the one we declared above: Given a path geometry on  $X$ , there is a canonical regular and normal Cartan geometry of type  $G/Q$  (according to the notation in 3.3) induced on the projectivized tangent bundle  $PTX$ . The notion of torsion-freeness of the path geometry is exactly the torsion-freeness of this induced Cartan connection. Let  $D \subset TPTX$  be the one-dimensional distribution corresponding the  $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$  as in (9) and let  $M$  be a local leaf space for this distribution. The elements in  $M$  locally parametrize the paths on  $X$ . In the torsion-free case, the Cartan connection on  $PTX$  descends to an integrable Grassmannian structure on  $M$  so that  $PTX \cong CM$ . The interpretation of the Grassmannian structure on  $M$  as a para-quaternionic one, together with the typical related constructs, might bring a new inside into the study of the geometry of second order ODE systems.

(2) There are another geometric structures related to the algebra of para-quaternions. In particular, we think of several well-established para-quaternionic subgeometries (i.e. cases that a para-quaternionic structure is defined only on a subbundle of the tangent bundle), among which the Lie contact structures play a prominent role. Lie contact structure appears naturally on the projectivized cotangent bundle of a conformal manifold, see e.g. subsections 4.2.5., 4.5.3. and 4.5.4. in [7] for some overview and detail. Some of the constructions above may be easily adapted and efficiently used for these structures. This is one of the intents of our forthcoming article.

## Appendix A. Some generalities on Cartan and parabolic geometries

In the above arguments we have mostly used just very standard facts from the theory of Cartan, respectively parabolic geometries. Here we briefly summarize what we need; the main signpost is again [7].

**A.1. Cartan geometry, Cartan curvature and torsion.** The notion of Cartan connections and Cartan geometries is the result of an attempt to put the original Cartan's ideas into the framework of fiber bundles and connections. The initial motivation was solving the equivalence problem for various geometric structures, later interpreted in terms of prolongations of principal frame bundles together with the construction of a (canonical) absolute parallelism. In this respect, the understanding of the model situation is substantial.

(1) For a Lie group  $G$  and its Lie subgroup  $H \subset G$ , the model Cartan geometry of type  $G/H$  is just the Klein geometry of the homogeneous space  $G/H$ . Let  $\mathfrak{g}$  be the Lie algebra of  $G$  and let  $G \rightarrow G/H$  be the natural  $H$ -principal bundle. The (left) Maurer–Cartan form  $\omega \in \Omega^1(G, \mathfrak{g})$  on the Lie group  $G$  has number of properties, some of whose are taken as the defining properties for general Cartan connections. Namely,

- (i) it provides the global identification  $TG \cong G \times \mathfrak{g}$ ,
- (ii) it is  $H$ -equivariant with respect to the principal right action on  $G$  and the adjoint action on  $\mathfrak{g}$ ,
- (iii) it reproduces the fundamental vector fields of the principal  $H$ -action.

It further satisfies the structure equation  $d\omega + \omega \wedge \omega = 0$ .

(2) The *Cartan geometry* of type  $G/H$  on a smooth manifold  $M$  consists of a principal  $H$ -bundle  $\mathcal{G} \rightarrow M$  together with a *Cartan connection*, which is a  $\mathfrak{g}$ -valued one-form on  $\mathcal{G}$  satisfying the properties (i)–(iii) above, provided that  $G$  is substituted by  $\mathcal{G}$  there. Note that Cartan connection is not a principal connection.

A morphism of two Cartan geometries of the same type is a morphism of the corresponding principal bundles that preserves the Cartan connection. The *Cartan curvature* of a Cartan geometry  $(\mathcal{G} \rightarrow M, \omega)$  is defined by

$$\kappa := d\omega + \omega \wedge \omega.$$

It is a two-form on  $\mathcal{G}$  with values in  $\mathfrak{g}$ , which is strictly horizontal, i.e. it vanishes under the insertion of any vertical vector. The Cartan curvature vanishes identically if and only if the Cartan geometry is locally isomorphic to its homogeneous model.

(3) For any Cartan geometry  $(\mathcal{G} \rightarrow M, \omega)$ , the Cartan connection provides an identification of the tangent bundle  $TM$  with the associate bundle to  $\mathcal{G}$  with the typical fibre  $\mathfrak{g}/\mathfrak{h}$  with respect to the action of  $H$  induced by the adjoint action on  $\mathfrak{g}$ . The latter bundle is denoted by  $\mathcal{G} \times_H (\mathfrak{g}/\mathfrak{h})$  and its elements are written as  $\llbracket u, X + \mathfrak{h} \rrbracket$ , where  $u \in \mathcal{G}$  and  $X \in \mathfrak{g}$ . By definition,

$$\llbracket u, X + \mathfrak{h} \rrbracket = \llbracket uh, \text{Ad}_{h^{-1}} X + \mathfrak{h} \rrbracket,$$

for any  $h \in H$ . In concrete terms, the identification  $\mathcal{G} \times_H (\mathfrak{g}/\mathfrak{h}) \cong TM$  may be stated as

$$\llbracket u, X + \mathfrak{h} \rrbracket \mapsto T_u \pi(\omega^{-1}(X)(u)),$$

where  $\pi$  is the bundle projection  $\mathcal{G} \rightarrow M$  and  $\omega^{-1}(X)$  denotes the *constant vector field* given by  $X \in \mathfrak{g}$ , which is the unique vector field on  $\mathcal{G}$  determined by  $\omega(\omega^{-1}(X)(u)) = X$ , for any  $u \in \mathcal{G}$ .

Under this identification, various geometric objects on  $M$  are reflected as  $H$ -invariant objects in  $\mathfrak{g}/\mathfrak{h}$ , e.g. an  $H$ -invariant subspace in  $\mathfrak{g}/\mathfrak{h}$  gives rise to a distribution in the tangent bundle  $TM$ . Furthermore, a vector field on  $M$  is represented by an  $H$ -equivariant map  $\mathcal{G} \rightarrow \mathfrak{g}/\mathfrak{h}$ , the so-called frame form. The same logic applies to any tensor field on  $M$ , more generally, to sections of any associated bundle to  $\mathcal{G}$ . In particular, the frame form of the Cartan curvature  $\kappa$  is an  $H$ -equivariant function

$$\mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}.$$

Composing with the quotient projection  $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$ , we obtain an  $H$ -equivariant map  $\mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes (\mathfrak{g}/\mathfrak{h})$ , which represents a tensor field  $\tau \in \Omega^2(M, TM)$ , which is called the *torsion* of the Cartan geometry  $(\mathcal{G} \rightarrow M, \omega)$ .

### A.2. Parabolic geometry, regularity, normality, and harmonic curvature.

*Parabolic geometry* is a Cartan geometry of type  $G/P$ , where  $G$  is a semi-simple Lie group and  $P \subset G$  is its parabolic subgroup. In contrast to general Cartan geometries, there is a canonical normalization condition for parabolic geometries, which we emphasize below. Let us start with a bit of notation.

(1) Let  $\mathfrak{p} \subset \mathfrak{g}$  be the Lie algebras corresponding to  $P \subset G$ . As a Lie algebra,  $\mathfrak{p}$  is a semi-direct sum of the nilpotent ideal  $\mathfrak{p}_+$  and a reductive part  $\mathfrak{g}_0$ . Let  $G_0$  be the Lie subgroup in  $P$  with the Lie algebra  $\mathfrak{g}_0$ . Defining  $P_+ := \exp \mathfrak{p}_+$ ,  $G_0$  is naturally identified with the factor group  $P/P_+$ . The subgroup  $P_+$  acts freely on the principal  $P$ -bundle  $\mathcal{G} \rightarrow M$ , hence the quotient bundle  $\mathcal{G}_0 := \mathcal{G}/P_+ \rightarrow M$  is a principal bundle with the structure group  $G_0$ .

Any parabolic subalgebra  $\mathfrak{p} \subset \mathfrak{g}$  is associated with a  $G_0$ -invariant grading of  $\mathfrak{g}$ ,

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k,$$

so that  $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ . The corresponding filtration on  $\mathfrak{g}$  is  $P$ -invariant and gives rise to a filtration of the tangent bundle  $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$  which is of length  $k$ . On the other hand, the associated graded bundle is denoted by  $\text{gr}(TM)$  and it is identified with the associated bundle  $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_-$ , where  $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ .

(2) There are two natural algebraic brackets on  $\text{gr}(TM)$ : one is induced by the Lie bracket of vector fields on  $TM$ , the other comes up from the bracket in the Lie algebra  $\mathfrak{g}_-$  due to the identification  $\text{gr}(TM) \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}_-$  above. The parabolic geometry is called *regular* if these two natural brackets coincide. The regularity can be characterized as follows: The Cartan–Killing form on  $\mathfrak{g}$  provides an identification of  $\mathfrak{p}_+$  with the dual space to  $\mathfrak{g}/\mathfrak{p}$ , which is equivariant with respect to the adjoint action of  $P$ . Hence the curvature function of a parabolic geometry of type  $G/P$  may be seen as a  $P$ -equivariant function

$$\mathcal{G} \rightarrow \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}.$$

Now, the grading of  $\mathfrak{g}$  induces a grading on the space  $\Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$ . The parabolic geometry is regular if and only if the components of negative homogeneity of the curvature function vanishes identically. Note that torsion-free parabolic geometries are automatically regular. Altogether, for a regular parabolic geometry, there is an underlying geometric structure on  $M$  consisting of a filtration of the tangent bundle (which is compatible with the Lie bracket of vector fields) and a reduction of the structure group of  $\text{gr}(TM)$  to the subgroup  $G_0$ .

(3) The correspondence between underlying geometric structures of a “parabolic type” and considered regular parabolic geometries can be made bijective imposing

a normalization condition. The parabolic geometry is called *normal* if the curvature function takes values in the kernel  $\ker \partial^* \subset \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$ , where

$$\partial^* : \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g} \rightarrow \mathfrak{p}_+ \otimes \mathfrak{g}$$

is the codifferential in the complex computing the Lie algebra homology of  $\mathfrak{p}_+$  with coefficients in  $\mathfrak{g}$ . For normal parabolic geometries, the composition of the curvature function  $\kappa$  with the quotient projection  $\ker \partial^* \rightarrow \ker \partial^* / \text{im } \partial^* = H_2(\mathfrak{p}_+, \mathfrak{g})$  yields a  $P$ -equivariant map  $\mathcal{G} \rightarrow H_2(\mathfrak{p}_+, \mathfrak{g})$ , the so-called *harmonic curvature* denoted by  $\kappa_H$ . In fact  $P_+$  acts trivially on  $H_2(\mathfrak{p}_+, \mathfrak{g})$ , which means the corresponding associated bundle allows an interpretation in terms of the underlying structure. Hence the harmonic curvature (or its components distinguished by the homogeneity degree) forms the fundamental invariant of the underlying geometric structure.

(4) As a  $G_0$ -module,  $\mathfrak{g}/\mathfrak{p}$  is isomorphic to  $\mathfrak{g}_-$  and, consequently,  $\Lambda^i \mathfrak{g}_-^* \otimes \mathfrak{g} \cong \Lambda^i \mathfrak{p}_+ \otimes \mathfrak{g}$ . The differential  $\partial$  in the complex computing the cohomology of  $\mathfrak{g}_-$  with coefficients in  $\mathfrak{g}$  turns out to be adjoint to  $\partial^*$  with respect to an appropriate inner product. This also gives rise to the  $G_0$ -equivariant self-adjoint endomorphism  $\square := \partial \circ \partial^* + \partial^* \circ \partial$ , the so-called Kostant Laplacian, which determines a  $G_0$ -invariant Hodge decomposition of the chain complex. Hence, the kernel of this operator,

$$\ker \square \subset \ker \partial^* \subset \Lambda^i \mathfrak{p}_+ \otimes \mathfrak{g},$$

is isomorphic to the homology group in each considered dimension. For regular and normal parabolic geometries, it follows that the lowest non-zero homogeneous component of  $\kappa$  has values in  $\ker \square \subset \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$ , i.e. it coincides with the corresponding homogeneous component of  $\kappa_H$ . In particular,  $\kappa$  vanishes identically if and only if  $\kappa_H$  does. Note that the  $G_0$ -representation  $\ker \square$  is algorithmically computable, hence the previous information finds a lot of very useful applications.

**A.3. Weyl connections and exact Weyl connections.** For any parabolic geometry, there is a canonical class of underlying affine connections on the base manifold generalizing the Weyl connections for conformal structures.

(1) Let  $(\mathcal{G} \rightarrow M, \omega)$  be a parabolic geometry of type  $G/P$  and let  $\mathcal{G}_0 = \mathcal{G}/P_+ \rightarrow M$  be the underlying principal  $G_0$ -bundle as above. A global smooth  $G_0$ -equivariant section  $\sigma : \mathcal{G}_0 \rightarrow \mathcal{G}$  of the canonical projection  $\mathcal{G} \rightarrow \mathcal{G}_0$  is called a *Weyl structure*. In particular, any Weyl structure provides a reduction of the  $P$ -principal bundle  $\mathcal{G} \rightarrow M$  to the subgroup  $G_0 \subset P$ . The pull-back  $\sigma^* \omega_0 \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$  of the  $\mathfrak{g}_0$ -component of the Cartan connection  $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$  defines a principal connection on the principal bundle  $\mathcal{G}_0$ , the *Weyl connection* of the Weyl structure  $\sigma$ . Any Weyl connection induces connections on all bundles associated to  $\mathcal{G}_0$ , which are also called by the same name. By definition, any Weyl connection preserves the underlying geometric structure on  $M$ . The family of Weyl connections is parametrized by one-forms on  $M$ .

(2) There are particularly convenient bundles associated to  $\mathcal{G}_0$  such that the induced connection from  $\sigma^* \omega_0$  is sufficient to determine whole the Weyl structure  $\sigma$ . These are the so-called scale bundles, which are the oriented line bundles over  $M$  defined as follows. Any principal  $\mathbb{R}_+$ -bundle associated to  $\mathcal{G}_0$  is determined by a group homomorphism  $\lambda : G_0 \rightarrow \mathbb{R}_+$  whose derivative is denoted by  $\lambda' : \mathfrak{g}_0 \rightarrow \mathbb{R}$ . Since the Lie algebra  $\mathfrak{g}_0$  is reductive, it splits into a direct sum of the center  $\mathfrak{z}(\mathfrak{g}_0)$  and the semi-simple part  $\mathfrak{g}_0^{ss}$ . Hence the only elements that can act non-trivially by  $\lambda'$  are from  $\mathfrak{z}(\mathfrak{g}_0)$ . The restriction of the Cartan–Killing form  $B$  on  $\mathfrak{g}$  to  $\mathfrak{g}_0$  and

further to  $\mathfrak{z}(\mathfrak{g}_0)$  is non-degenerate. Altogether, for any representation  $\lambda' : \mathfrak{g}_0 \rightarrow \mathbb{R}$  there is a unique element  $E_\lambda \in \mathfrak{z}(\mathfrak{g}_0)$  such that

$$\lambda'(A) = B(E_\lambda, A) \quad (27)$$

for all  $A \in \mathfrak{g}_0$ . An element  $E_\lambda \in \mathfrak{z}(\mathfrak{g}_0)$  is called a *scaling element* if it acts by a non-zero real scalar on each  $G_0$ -irreducible component of  $\mathfrak{p}_+$ . (Scaling elements always exist, e.g. the grading element of  $\mathfrak{g}$  is scaling element.) A *scale bundle* is a principal  $\mathbb{R}_+$ -bundle associated to  $\mathcal{G}_0$  via a homomorphism  $\lambda : G_0 \rightarrow \mathbb{R}_+$ , whose derivative is given by (27) for some scaling element  $E_\lambda$ . The scale bundle  $\mathcal{L} \rightarrow M$  corresponding to  $\lambda$  is naturally identified with the quotient bundle  $\mathcal{G}_0 / \ker \lambda \rightarrow M$ .

Now it turns out, the principal connections on a scale bundle  $\mathcal{L} \rightarrow M$  are in a bijective correspondence with the Weyl structures  $\sigma : \mathcal{G}_0 \rightarrow \mathcal{G}$ . In particular, any section of the scale bundle gives rise to a flat principal connection on  $\mathcal{L}$ , the corresponding Weyl structure is called *exact*. Due to the identification  $\mathcal{L} = \mathcal{G}_0 / \ker \lambda$ , sections of the scale bundle correspond to reductions of the principal bundle  $\mathcal{G}_0 \rightarrow M$  to the structure group  $\ker \lambda \subset G_0$ . Altogether, exact Weyl connections have holonomies contained in  $\ker \lambda$ . Of course, the exact Weyl connection determined by a section of the scale bundle preserves the geometric quantity corresponding to this section.

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